Partial Serre Duality and Cocompact Objects

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Outline

The talk is based on joint work with Steffen Oppermann and Torkil Stai:

- *Change of Rings and Singularity Categories*, Adv. Math. 350 (2019), 190–241.
- *Partial Serre duality and cocompact objects*, coming soon.

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The plan of the talk is as follows:

1. Motivation
2. Cocompact objects
3. Partial Serre duality
4. Examples

Motivation

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Theorem (Beligiannis-Reiten 2007, Aihara-Iyama 2012)

Let $\mathcal{T}$ be a triangulated category with coproducts and let $\mathcal{X}$ be a set of compact objects. Then the pair

$$\left({}^{\perp_0}(\mathcal{X}^{\perp_{\leq 0}}), \mathcal{X}^{\perp_{\leq 0}}\right)$$

is a t-structure in $\mathcal{T}$.

Notation: $\mathcal{X}^{\perp_{\leq 0}} = \{Y \in \mathcal{T} \mid \forall i \leq 0 : \mathrm{Hom}_{\mathcal{T}}(\mathcal{X}, Y[i]) = 0\}$.

For all objects $T \in \mathcal{T}$ there is a triangle

$$T_{\mathcal{X}} \to T \to T_{\mathcal{X}^{\perp_{\leq 0}}} \to T_{\mathcal{X}}[1]$$

with $T_{\mathcal{X}} \in {}^{\perp_0}(\mathcal{X}^{\perp_{\leq 0}})$ and $T_{\mathcal{X}^{\perp_{\leq 0}}} \in \mathcal{X}^{\perp_{\leq 0}}$.

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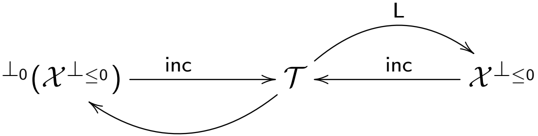

$$T \mapsto \mathsf{L}(T) := T_{\mathcal{X}^{\perp_{\leq 0}}}$$

The proof of the above Theorem gives an “explicit” construction of L. The functor L assigns to each object T of $\mathcal{T}$ the homotopy colimit of the cones of iterated right Add$\mathcal{X}$-approximations of T.
Note that Add$\mathcal{X}$ is the subcategory consisting of summands of coproducts of objects in $\mathcal{X}$.

Let Λ be an Artin algebra. We write $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda)$ for the homotopy category of acyclic complexes of injective Λ-modules. It is a compactly generated triangulated category and

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Theorem (Krause 2005)

There is a triangle equivalence (up to direct summands):

$$K_{ac}(\mathrm{Inj}\Lambda)^c \simeq D_{sg}(\Lambda)$$

$D_{sg}(\Lambda) = D^b(\mathrm{mod}\Lambda)/K^b(\mathrm{proj}\Lambda)$: the singularity category of Λ (introduced by Buchweitz and Orlov).

Motivated by Krause's result, $K_{ac}(\mathrm{Inj}\Lambda)$ is called the big singularity category of Λ.

We have $\mathsf{K}_{ac}(\mathsf{Inj}\Lambda) = (\mathsf{i}\Lambda)^{\perp}$ in $\mathsf{K}(\mathsf{Inj}\Lambda)$, where the injective resolution $\mathsf{i}\Lambda$ of Λ is compact in $\mathsf{K}(\mathsf{Inj}\Lambda)$ ([Krause, Iyengar-Krause]).

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$$\big({}^{\perp_0}((\mathsf{i}\Lambda)^{\perp}), (\mathsf{i}\Lambda)^{\perp}\big)$$

is a stable t-structure in $\mathsf{K}(\mathsf{Inj}\Lambda)$. We have:

$$(\mathsf{i}\Lambda)^{\perp} = \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda) \overset{\xleftarrow{\quad\mathsf{L}\quad}}{\xrightarrow{\quad\mathsf{inc}\quad}} \mathsf{K}(\mathsf{Inj}\Lambda)$$

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$$(\mathrm{i}\Lambda)^{\perp} = \mathsf{K}_{\mathrm{ac}}(\mathrm{Inj}\Lambda) \underset{\mathrm{inc}}{\overset{\mathsf{L}}{\leftrightarrows}} \mathsf{K}(\mathrm{Inj}\Lambda)$$

Let $T_0 = T \in \mathsf{K}(\mathrm{Inj}\Lambda)$. Pick a right $\mathrm{Add}(\mathrm{i}\Lambda)$-approximation $X_0 \to T_0$ and complete it to a triangle

$$X_0 \to T_0 \to T_1 \to X_0[1].$$

Pick a right $\mathrm{Add}(\mathrm{i}\Lambda)$-approximation $X_1 \to T_1$ and complete it to a triangle

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We iterate in this fashion and then

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<u>Question</u>: Can we get R in a "dual explicit" way?

$$(\mathsf{i}\Lambda)^{\perp} = \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda) \xrightarrow{\ \mathsf{inc}\ } \mathsf{K}(\mathsf{Inj}\Lambda), \qquad \mathsf{L}, \mathsf{R} \colon \mathsf{K}(\mathsf{Inj}\Lambda) \to \mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda)$$

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Question: Can we get R in a "dual explicit" way?

$$(\mathrm{i}\Lambda)^{\perp} = \mathrm{K}_{\mathrm{ac}}(\mathrm{Inj}\Lambda) \begin{array}{c} \xleftarrow{\quad\mathrm{L}\quad} \\ \xrightarrow{\quad\mathrm{inc}\quad} \\ \xleftarrow[\quad\mathrm{R}\quad]{} \end{array} \mathrm{K}(\mathrm{Inj}\Lambda)$$

Naive idea: Cocompact objects, i.e. objects such that their contravariant Hom-functors sends coproducts to products.

0-Cocompact Objects

A sequence

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \cdots$$

of objects and morphisms in an abelian category is *dual Mittag-Leffler* if for each i the increasing sequence $0 \subset \mathrm{Ker} f_i \subset \mathrm{Ker} f_{i+1} f_i \subset \cdots$ stabilizes.

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Definition

An object X in a triangulated category $\mathcal{T}$ admitting products is 0-*cocompact* if $\mathrm{Hom}_{\mathcal{T}}(\mathrm{holim}\, Y_i, X) = 0$ for each sequence

$$\cdots \to Y_2 \to Y_1 \to Y_0$$

in $\mathcal{T}$ with the property that the induced sequence

$$\mathrm{Hom}_{\mathcal{T}}(Y_0, X[1]) \to \mathrm{Hom}_{\mathcal{T}}(Y_1, X[1]) \to \mathrm{Hom}_{\mathcal{T}}(Y_2, X[1]) \to \cdots$$

is dual Mittag-Leffler and $\mathrm{colim}\mathrm{Hom}_{\mathcal{T}}(Y_i, X) = 0$.

Example

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Remark

Let X_i be a set-indexed collection of 0-cocompact objects. Then the product $\prod X_i$ is 0-cocompact.

Remark (Why "0-cocompact"?)

An object $X \in \mathcal{T}$ is cocompact if for any set of objects $\{Y_i\}_{i \in I}$ in $\mathcal{T}$ we have an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\prod_{i \in I} Y_i, X) \cong \coprod_{i \in I} \mathrm{Hom}_{\mathcal{T}}(Y_i, X).$$

Let Λ be a finite dimensional algebra. Clearly, Λ is a compact object in $\mathrm{D}(\mathrm{Mod}\Lambda)$. On the other hand, the dual $\mathrm{D}(\Lambda)$ fails to be cocompact in the derived category. But $\mathrm{D}(\Lambda)$ is in fact 0-cocompact in $\mathrm{K}(\mathrm{Mod}\Lambda)$.

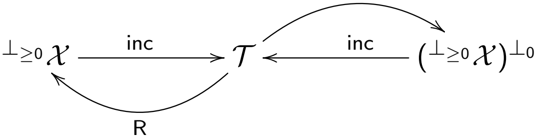
Replacing products by homotopy limits and coproducts by colimits, X being 0-cocompact simply means that if the right hand side of the above isomorphism is zero and some extra conditions hold, then the left hand side is also zero. It turns out that this property suffices for our purposes.

Theorem (Oppermann-P.-Stai 2019)

Let $\mathcal{T}$ be a triangulated category which has products. Let $\mathcal{X}$ be a set of 0-cocompact objects.

1. The pair
$$\left({}^{\perp_{\geq 0}}\mathcal{X}, ({}^{\perp_{\geq 0}}\mathcal{X})^{\perp_0}\right)$$
is a co-t-structure in $\mathcal{T}$.
2. The pair
$$\left({}^{\perp}\mathcal{X}, ({}^{\perp}\mathcal{X})^{\perp}\right)$$
is a stable t-structure. Moreover, $({}^{\perp}\mathcal{X})^{\perp}$ is the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{X}$ and is closed under products.

Picture:

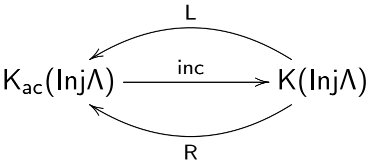


The proof of the above Theorem gives an "explicit" construction of R. The functor R assigns to each object T of $\mathcal{T}$ the homotopy limit of the cones of iterated right Prod$\mathcal{X}$-approximations of T.

Note that Prod$\mathcal{X}$ denotes the subcategory of summands of products of objects in $\mathcal{X}$.

Back to the big singularity category

We have:



Observe that $\mathsf{K}_{\mathsf{ac}}(\mathsf{Inj}\Lambda) = {}^{\perp}(\mathsf{D}(\Lambda))$ in $\mathsf{K}(\mathsf{Inj}\Lambda)$ and $\mathsf{D}(\Lambda)$ is 0-cocompact in $\mathsf{K}(\mathsf{Inj}\Lambda)$. Then

$$({}^{\perp}(\mathsf{D}(\Lambda)), ({}^{\perp}(\mathsf{D}(\Lambda)))^{\perp})$$

is a stable t-structure in $\mathsf{K}(\mathsf{Inj}\Lambda)$ and therefore we obtain the functor R:

Let $T_0 = T \in \mathsf{K}(\mathsf{Inj}\Lambda)$. Pick a left $\mathsf{ProdD}(\Lambda)$-approximation $T_0 \to Y_0$ and complete it to a triangle

$$T_1 \to T_0 \to Y_0 \to T_1[1].$$

Pick a left $\mathsf{ProdD}(\Lambda)$-approximation $T_1 \to Y_1$ and complete it to a triangle

$$T_2 \to T_1 \to Y_1 \to T_2[1].$$

We iterate in this fashion and then

$$\mathsf{R}(T) = \operatorname{holim} T_i.$$

Partial Serre Duality

Let $\mathcal{T}$ be a triangulated k-category for some commutative ring k. We choose an injective cogenerator I of $\mathrm{Mod}k$ and write $(-)^* = \mathrm{Hom}_k(-, I)$.

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Definition

A **partial Serre functor** for a subcategory $\mathcal{X} \subset \mathcal{T}$ is a functor $\mathbb{S} \colon \mathcal{X} \to \mathcal{T}$ such that

$$\operatorname{Hom}_{\mathcal{T}}(X, T)^* \cong \operatorname{Hom}_{\mathcal{T}}(T, \mathbb{S}X)$$

naturally in $X \in \mathcal{X}$ and in $T \in \mathcal{T}$.

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Theorem (Oppermann-P.-Stai)

Suppose $\mathcal{T}$ is a triangulated category, and let $\mathcal{X}$ be the full subcategory of all objects X such that $\mathrm{Hom}_{\mathcal{T}}(X, -)^*$ is representable. Then $\mathcal{X}$ is a triangulated subcategory of $\mathcal{T}$, and there is a partial Serre functor $\mathbb{S}\colon \mathcal{X} \to \mathcal{T}$. Moreover, $\mathbb{S}$ is a triangle functor.

Theorem (Oppermann-P.-Stai)

Let $\mathcal{T}$ be a triangulated category. If $X, Y \in \mathcal{T}$ satisfy

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then X is compact and Y is 0-cocompact.

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then X is compact and Y is 0-cocompact.

Sketch of Proof: We show that Y is 0-cocompact. Let

$$\mathbb{T}\colon \cdots \to T_2 \to T_1 \to T_0$$

be a sequence such that $\mathrm{colim}\mathrm{Hom}_{\mathcal{T}}(\mathbb{T}, Y) = 0$ and $\mathrm{Hom}_{\mathcal{T}}(\mathbb{T}, Y[1])$ is dual ML. We want to conclude that $\mathrm{Hom}_{\mathcal{T}}(\mathrm{holim}\mathbb{T}, Y) = 0$. Since $\mathrm{Hom}_{\mathcal{T}}(\mathrm{holim}\mathbb{T}, Y) \cong \mathrm{Hom}_{\mathcal{T}}(X, \mathrm{holim}\mathbb{T})^*$ it suffices to show that $\mathrm{Hom}_{\mathcal{T}}(X, \mathrm{holim}\mathbb{T}) = 0$. There is a short exact sequence

$$0 \to \lim^1\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}[-1]) \to \mathrm{Hom}_{\mathcal{T}}(X, \mathrm{holim}\mathbb{T}) \to \lim\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}) \to 0,$$

and it suffices to show that the outer terms vanish.

Since the system $\mathrm{Hom}_{\mathcal{T}}(\mathbb{T}, Y[1])$ is the dual of the system $\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}[-1])$, it follows that the latter is ML, and therefore its derived limit vanishes.
Also, we have $0 = \mathrm{colim}\mathrm{Hom}_{\mathcal{T}}(\mathbb{T}, Y) = \mathrm{colim}(\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T})^*)$. Then

$$\mathrm{colim}(\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T})^*)^* \cong \lim(\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T})^{**}) = 0$$

Since lim is left exact, the monomorphism of diagrams $\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}) \to \mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T})^{**}$ induces a monomorphism

$$\lim\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}) \to \lim\left(\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T})^{**}\right)$$

and therefore $\lim\mathrm{Hom}_{\mathcal{T}}(X, \mathbb{T}) = 0$.

We now show that X is compact. Let $\{T_i\}$ be a set of objects in $\mathcal{T}$ and for each i let $\mu_i \colon T_i \to \coprod T_i$ be the canonical morphism. By assumption we have

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{T}}(X, \coprod T_i)^* & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{T}}(\coprod T_i, Y) \\ \downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{T}}(X,\mu_i)^*} & & \downarrow{\scriptstyle \mu_i^*} \\ \operatorname{Hom}_{\mathcal{T}}(X, T_i)^* & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{T}}(T_i, Y) \end{array}$$

and taking the product in the lower row gives the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{T}}(X, \coprod T_i)^* & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{T}}(\coprod T_i, Y) \\ \downarrow & & \downarrow \\ \prod \operatorname{Hom}_{\mathcal{T}}(X, T_i)^* & \xrightarrow{\cong} & \prod \operatorname{Hom}_{\mathcal{T}}(T_i, Y) \end{array}$$

The left hand vertical morphism is the dual of the natural morphism

$$\coprod \operatorname{Hom}_{\mathcal{T}}(X, T_i) \to \operatorname{Hom}_{\mathcal{T}}(X, \coprod_i T_i),$$

so the claim follows since $(-)^*$ reflects isomorphisms. □

Remark

The above Theorem says that if $\mathbb{S}\colon \mathcal{X} \to \mathcal{T}$ is a partial Serre functor, then $\mathcal{X}$ consists of compact objects while the essential image $\mathbb{S}(\mathcal{X})$ consists of 0-cocompact objects.

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Corollary

Let $\mathcal{T}$ be a compactly generated triangulated category. Then there is a partial Serre functor $\mathbb{S}\colon \mathcal{T}^{\mathsf{c}} \to \mathcal{T}$, and the essential image $\mathbb{S}(\mathcal{T}^{\mathsf{c}})$ is a set of 0-cocompact objects for $\mathcal{T}$.

Proof: By Brown representability, the functor $\mathrm{Hom}_{\mathcal{T}}(X, -)^*$ is representable for all X in $\mathcal{T}^{\mathsf{c}}$. Then there is a partial Serre functor $\mathbb{S}\colon \mathcal{T}^{\mathsf{c}} \to \mathcal{T}$ and the set $\mathbb{S}(\mathcal{T}^{\mathsf{c}})$ consists of 0-cocompact objects. □

0-cocompacts vs pure injectives

Let $\mathcal{T}$ be a compactly generated triangulated category. A triangle $X \to Y \to Z \to X[1]$ is called *pure* if for each compact object C, the induced sequence $0 \to \mathrm{Hom}_{\mathcal{T}}(C, X) \to \mathrm{Hom}_{\mathcal{T}}(C, Y) \to \mathrm{Hom}_{\mathcal{T}}(C, Z) \to 0$ is exact. An object $E \in \mathcal{T}$ is *pure-injective* if $\mathrm{Hom}_{\mathcal{T}}(-, E)$ takes pure triangles to short exact sequences.

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Remark

Each 0-cocompact object which arises from partial Serre duality is pure-injective.

Proof: Let $\mathbb{S}\colon \mathcal{X} \to \mathcal{T}$ a partial Serre functor. We claim that $\mathbb{S}X$ is pure-injective for each $X \in \mathcal{X}$. We only need to check that $\mathrm{Hom}_{\mathcal{T}}(-, \mathbb{S}X) \cong \mathrm{Hom}_{\mathcal{T}}(X, -)^*$ takes pure triangles to short exact sequences. As X is compact, the functor $\mathrm{Hom}_{\mathcal{T}}(X, -)$ does enjoy this property by definition. Since $(-)^*$ is exact, the claim follows. □

We do not know if 0-cocompactness in general imples pure injectivity.

There are pure-injective objects which are not 0-cocompact:

Example

The object $\mathbb{Q}$ is pure-injective in $\mathrm{D}(\mathrm{Mod}\mathbb{Z})$ but it is not 0-cocompact. We show that if an object X is 0-cocompact, then the subcategory ${}^{\perp}X$ is closed under countable products. Then we can check that ${}^{\perp}\mathbb{Q}$ does not enjoy this property.

Examples of Partial Serre Functors: Example 1

Let R be a ring. Let $M \in \mathrm{C}^{\mathrm{b}}(\mathrm{mod}R)$ and take a projective presentation

$$P_1 \xrightarrow{p} P_0 \to M \to 0$$

in $\mathrm{C}^{\mathrm{b}}(\mathrm{mod}R)$. Note that $P_1, P_0 \in \mathrm{C}^{\mathrm{b}}(\mathrm{proj}R)$ and are moreover contractible. Define

$$\mathbb{S}(M) = \mathrm{Ker}(\nu(p))[2]$$

where $\nu = (\mathrm{Hom}_R(-, R))^*$.

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Theorem (OPS)

The functor $\mathbb{S} \colon \mathrm{K}^{\mathrm{b}}(\mathrm{mod}R) \to \mathrm{K}(\mathrm{Mod}R)$ is a partial Serre functor, i.e.

$$\mathrm{Hom}_{\mathrm{K}}(M, X)^* \cong \mathrm{Hom}_{\mathrm{K}}(X, \mathbb{S}M)$$

naturally in $M \in \mathrm{K}^{\mathrm{b}}(\mathrm{mod}R)$ and $X \in \mathrm{K}(\mathrm{Mod}R)$.

Example 2

Let R be a ring. By Neeman (2014) the inclusion functor

$$\mathsf{K}(\mathrm{Inj}R) \to \mathsf{K}(\mathrm{Mod}R)$$

has a left adjoint denoted by λ. By Krause (2005), If R is noetherian then taking injective resolutions identifies $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}R)$ with the full subcategory $\mathsf{K}(\mathrm{Inj}R)^{\mathrm{c}}$ of compact objects in $\mathsf{K}(\mathrm{Inj}R)$.

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Theorem (OPS)

Let R be a noetherian ring. The functor $\mathbb{S}\colon \mathsf{K}(\mathrm{Inj}R)^{\mathsf{c}} \to \mathsf{K}(\mathrm{Inj}R)$ given by $\mathbb{S}(\lambda X) = \nu(\rho X)$ for each $X \in \mathsf{D}^{\mathsf{b}}(\mathrm{mod}R)$, is a partial Serre functor.

Note that ρX is a projective resolution of X with finitely generated terms and $\nu = (\mathrm{Hom}_R(-, R))^*$.

Example 3

Theorem (OPS)

Let R be a noetherian ring. There is a partial Serre functor

$$\mathbb{S}' \colon \mathrm{D_{sg}}(R) \to \mathrm{K_{ac}}(\mathrm{Inj}R)$$

given as $\mathbb{S}'(X) = \mathrm{R}\nu\rho(X)$ for $X \in \mathrm{D^b}(\mathrm{mod}R)$.

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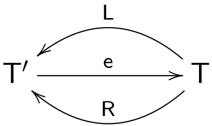
Proof: The result follows by the next two observations:

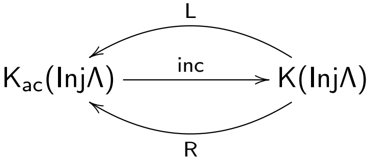
First step: Consider an adjoint triple of triangle functors



If $\mathbb{S} \colon \mathcal{X} \to \mathcal{T}$ is a partial Serre functor for $\mathcal{X} \subset \mathcal{T}$, then the essential image $\mathsf{L}(\mathcal{X}) \subset \mathcal{T}'$ admits a partial Serre functor $\mathbb{S}' \colon \mathsf{L}(\mathcal{X}) \to \mathcal{T}'$ given by $\mathbb{S}'(\mathsf{L}X) = \mathsf{R}\mathbb{S}X$ for each $X \in \mathcal{X}$:

$$\mathrm{Hom}_{\mathsf{T}'}(\mathsf{L}X, -)^* \cong \mathrm{Hom}_{\mathsf{T}}(X, e-)^* \cong \mathrm{Hom}_{\mathsf{T}}(e-, \mathbb{S}X) \cong \mathrm{Hom}_{\mathsf{T}'}(-, \mathsf{R}\mathbb{S}X).$$

Second Step: There is a partial Serre functor $\mathbb{S}\colon \mathsf{K}(\mathrm{Inj}R)^{\mathsf{c}} \to \mathsf{K}(\mathrm{Inj}R)$ by Example 2 and we have the adjoint triple:



□

Example 4

Let Λ be an Artin algebra, we write D for the standard duality. Recall that the compacts of $\mathrm{D}(\mathrm{Mod}\Lambda)$ are the perfect complexes $\mathrm{perf}\Lambda$. Then the functor

$$\mathbb{S} = - \otimes^{\mathbb{L}}_{\Lambda} \mathrm{D}\Lambda \colon \mathrm{perf}\Lambda \to \mathrm{D}(\mathrm{Mod}\Lambda)$$

is a partial Serre functor inducing an equivalence

$$\mathrm{perf}\Lambda \simeq \mathbb{S}(\mathrm{perf}\Lambda) = \{\text{bounded complexes over } \mathrm{inj}\Lambda\}$$

of subcategories of $\mathrm{D}(\mathrm{Mod}\Lambda)$.
Note that $\mathbb{S}$ is an autoequivalence on $\mathrm{perf}\Lambda$ if and only if Λ is Gorenstein, in which case each perfect complex is 0-cocompact in $\mathrm{D}(\mathrm{Mod}\Lambda)$.

Thank you!

Thank you for your attention!