Exact structures and degeneration of Hall algebras

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Hall algebras

Fix $k = \mathbb{F}_q$. Let $\mathcal{C}$ be a small $k-$linear abelian category such that

$$|\operatorname{Hom}(A,B)| < \infty, \quad |\operatorname{Ext}^1(A,B)| < \infty, \qquad \forall A, B \in \mathcal{C}.$$

Definition-Theorem (Ringel)

The Hall algebra $\mathcal{H}(\mathcal{C})$ is the $\mathbb{Q}-$algebra with a basis $\{u_X \mid X \in \operatorname{Iso}(\mathcal{C})\}$ and multiplication

$$u_A * u_C = \sum_{B \in \operatorname{Iso}(\mathcal{C})} \frac{|\operatorname{Ext}^1(A,C)_B|}{|\operatorname{Hom}(A,C)|} u_B.$$

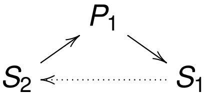
$\mathcal{H}(\mathcal{C})$ is associative and unital. It is usually not $q-$commutative.

Here $\operatorname{Ext}^1(A,C)_B \subset \operatorname{Ext}^1(A,C)$ is given by short exact sequences

$$C \rightarrowtail B' \twoheadrightarrow A$$

with $B' \xrightarrow{\sim} B$.

Example: $\operatorname{mod} kA_2 = \operatorname{mod} k(1 \longrightarrow 2)$



$$u_{S_2} * u_{S_1} = u_{S_1 \oplus S_2};$$
$$u_{S_1} * u_{S_2} = u_{S_1 \oplus S_2} + (q-1) u_{P_1}.$$

$$u_{P_1} = \frac{1}{q-1}[u_{S_1}, u_{S_2}]. \qquad (1)$$

$$\mathfrak{g}(A_2) = sl_3; \qquad \mathfrak{n}^+(A_2) = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Gabriel : $\alpha_1 \mapsto S_1, \ \alpha_2 \mapsto S_2, \ \alpha_1 + \alpha_2 \mapsto P_1.$

Ringel : $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right] \rightsquigarrow (1)$

Theorem (Ringel-Green)

Let Q be a finite acyclic quiver. Then there is a Hopf algebra map

$$U_{\sqrt{q}}(\mathfrak{b}^-(Q)) \hookrightarrow \mathcal{H}^{ex}_{tw}(\operatorname{mod} kQ).$$

This is an isomorphism if and only if Q is of Dynkin type.

- $U_{\sqrt{q}}(\mathfrak{b}^-(Q))$ is the Borel part of the quantized Kac-Moody algebra associated to Q.
- $\mathcal{H}^{ex}_{tw}(\operatorname{mod} kQ)$ is $\mathcal{H}(\operatorname{mod} kQ)$ extended by $\mathbb{Q}K_0(\operatorname{mod} kQ)$, with the multiplication twisted by the square root of the Euler form (one should consider it over $\mathbb{Q}(\sqrt{q})$). It has a Hopf algebra structure.

Green and Xiao endowed the (twisted extended) Hall algebra of any **hereditary abelian** category with a Hopf algebra structure.

Exact structures

Quillen: *Exact categories*. Axiomatize extension-closed subcategories of abelian categories.

Examples

- The full subcategory of projective objects in an abelian category.
- Category of vector bundles on a scheme.
- Torsion and torsion free subcategories of abelian categories.

Theorem (Hubery)

Let $\mathcal{E}$ be a $\mathrm{Hom}-$ and Ext^1-finite, $k-$linear small exact category. The Hall algebra $\mathcal{H}(\mathcal{E})$ defined in the same way is associative and unital.

Exact structures II

Axiomatics suggests that an additive category may admit many different exact structures: one can choose different classes of *admissible short exact sequences* (= *conflations*).
Let $(\mathcal{A}, \mathcal{E})$ be an additive category endowed with an exact structure.
Then $\mathrm{Ext}^1_{\mathcal{E}}(-,-) : \mathcal{A}^{op} \times \mathcal{A} \to \mathbf{Ab}$ is an additive bifunctor.
Upshot: $\mathcal{E}$ is uniquely determined by $\mathrm{Ext}^1_{\mathcal{E}}(-,-)$.

- Any *extension-closed* full subcategory of $(\mathcal{A}, \mathcal{E})$ has an induced exact structure (with the same $\mathrm{Ext}^1_{\mathcal{E}}(-,-)$).
- Any *closed* additive sub-bifunctor $\mathbb{F} \subset \mathrm{Ext}^1(-,-)$ defines a “smaller”, or *relative*, exact structure on $\mathcal{A}$. This is equivalent to taking a sub-class of conflations (satisfying Quillen’s axioms).
- Any localization with respect to a *right filtering* exact subcategory has an induced exact structure.

NB: Some natural quotients and localizations of exact categories have no induced exact structures. More on that later...

Hall algebras II

The Hall algebra of an exact category depends not only on the underlying additive category. It depends on the choice of exact structure!

Example

- Ringel-Green: $\mathcal{H}_{tw}(\operatorname{mod} kQ, \mathrm{ab}) \overset{\sim}{\leftarrow} U_{\sqrt{q}}(\mathfrak{n}^+)$.
- For any additive category $\mathcal{A}$, the Hall algebra $\mathcal{H}(\mathcal{A}, \mathrm{add})$ of the split exact structure is a polynomial algebra in $q-$commuting variables.
- $\mathcal{H}(\operatorname{mod} kA_2, \mathrm{add})$ is the polynomial algebra in u_{S_1}, u_{S_2}, and u_{P_1}, modulo relations:

$$u_{S_2} * u_{S_1} = u_{S_1 \oplus S_2} = u_{S_1} * u_{S_2};$$

$$u_{S_1} * u_{P_1} = u_{S_1 \oplus P_1} = \frac{1}{q} u_{P_1} * u_{S_1};$$

$$u_{S_2} * u_{P_1} = q u_{S_2 \oplus P_1} = q u_{P_1} * u_{S_2}.$$

Degree functions and filtrations

Definition

Consider a function $w : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}$. We say that w is

- *additive* if $w(M \oplus N) = w(M) + w(N)$ for all M and N;
- an $\mathcal{E}-$*quasi-valuation* if $w(X) \leq w(M \oplus N)$ whenever there exists a conflation $N \rightarrowtail X \twoheadrightarrow M$ in $\mathcal{E}$.
- an $\mathcal{E}-$*valuation* if it is an additive $\mathcal{E}-$quasi-valuation.

If $\mathcal{A}$ is Krull-Schmidt, an additive function is the same as a function on indecomposables: $\mathrm{Ind}(\mathcal{A}) \to \mathbb{N}$. Suppose $\mathcal{A}$ is $\mathrm{Hom}-$ finite.

Example

- $w_X := \dim \mathrm{Hom}(X, -)$ is a valuation for any exact structure on $\mathcal{A}$.. If X is $\mathcal{E}-$projective, it is additive on conflations in $\mathcal{E}$.
- $\dim \mathrm{End}(-)$ is a quasi-valuation for any exact structure on $\mathcal{A}$. But it is usually not additive.

Main Theorems

Let $\mathcal{A}$ be a Hom $-$finite $k-$linear idempotent complete additive category. Let $\mathcal{E}$ be an Ext^1 $-$finite exact structure on $\mathcal{A}$.

Theorem I (F.-G.)

Each $\mathcal{E}-$valuation $w : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}$ induces a filtration $\mathcal{F}_w$ on $\mathcal{H}(\mathcal{E})$. The associated graded is $\mathcal{H}(\mathcal{E}')$ for a smaller exact structure $\mathcal{E}' \leq \mathcal{E}$ on $\mathcal{A}$.

$\mathcal{A}$ is *locally finite* if $\forall X \in \mathcal{A}$, there exists only finitely many $Y, Z \in \mathrm{Ind}(\mathcal{A})$ s.t. $\quad \mathrm{Hom}(X, Y) \neq 0, \quad \mathrm{Hom}(Z, X) \neq 0.$

Theorem II (F.-G.)

Suppose $\mathcal{A}$ is locally finite. Then for each exact substructure $\mathcal{E}' < \mathcal{E}$, there exists an $\mathcal{E}-$valuation w such that

$$\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}').$$

As w, one can take a (formal) sum of $\dim(\mathrm{Hom}(X, -))$.

Lattice of exact structures I

Exact structures on an additive category form a poset.

Theorem (Brüstle-Hassoun-Langford-Roy)

This is a bounded complete lattice.

For any conflation $\delta : A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C$ in $\mathcal{E}$, one has an exact sequence of right $\mathcal{A}-$modules $\mathcal{A}^{op} \to \mathbf{Ab}$

$$0 \to \mathrm{Hom}(-, A) \xrightarrow{\mathrm{Hom}(-,f)} \mathrm{Hom}(-, B) \xrightarrow{\mathrm{Hom}(-,g)} \mathrm{Hom}(-, C).$$

The *contravariant defect* of δ is $\mathrm{Coker}(\mathrm{Hom}(-, g))$.
The category $\mathbf{def}\,\mathcal{E}$ of contravariant defects of conflations in $\mathcal{E}$ is an abelian category. Its simple objects are the defects of *Auslander-Reiten* (= *almost split*) conflations.
If $\mathcal{A}$ is Krull-Schmidt and locally finite, each object in $\mathbf{def}\,\mathcal{E}$ (for each $\mathcal{E}$!) has finite length.

Lattice of exact structures II

Theorem (..., Buan, Rump, Enomoto, F.-G.)

Each additive category $\mathcal{A}$ admits a unique maximal exact structure $(\mathcal{A}, \mathcal{E}^{\mathsf{max}})$. There is a lattice isomorphism between

- *The lattice of exact structures on $\mathcal{A}$;*
- *The lattice of Serre subcategories of the category* $\mathbf{def}(\mathcal{A}, \mathcal{E}^{\mathsf{max}})$.

If $\mathcal{A}$ is locally finite, these lattices are Boolean: they are isomorphic to the power set of $\mathrm{AR}-$*conflations of* $\mathcal{E}^{\mathsf{max}}$.

Sketches of the proofs

Proof of Theorem I

Each $\mathcal{E}-$valuation w induces a function $\widetilde{w}: \mathrm{Iso}(\mathbf{def}\,\mathcal{E}) \to \mathbb{N}$. This function is additive on short exact sequences.
Then $\mathrm{Ker}(\widetilde{w})$ is a Serre subcategory of $\mathbf{def}\,\mathcal{E}$. So it defines an exact substructure $\mathcal{E}' \leq \mathcal{E}$. Then $\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')$.

Proof of Theorem II

Let $\mathrm{Ex}_+(\mathcal{E})$ be a sub-semigroup of $K_0^{\mathrm{add}}(\mathcal{A})$ generated by alternating sums $[X] - [Y] + [Z]$ for all conflations $X \rightarrowtail Y \twoheadrightarrow Z$.
Let $\mathrm{AR}_+(\mathcal{E})$ be its sub-semigroup generated by alternating sums for all $\mathrm{AR}-$conflations.
If $\mathcal{A}$ is locally finite, then $\mathrm{Ex}_+(\mathcal{E}) = \mathrm{AR}_+(\mathcal{E})$ for each exact structure $\mathcal{E}$ on $\mathcal{A}$. Using this, we can prove that $\mathbf{gr}_{\mathcal{F}_w}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')$, for

$$w := \sum_{X \in \mathrm{Ind}(\mathrm{proj}(\mathcal{E}')) \setminus \mathrm{Ind}(\mathrm{proj}(\mathcal{E}))} \dim \mathrm{Hom}(X, -).$$

Cones

Assume that $\mathcal{A}$ has finitely many indecomposables. Consider $\Lambda^{\mathcal{E},\mathcal{E}'} := \operatorname{Ker}(K_0(\mathcal{E}') \twoheadrightarrow K_0(\mathcal{E}))$. Let

$$\mathcal{C}^{\mathcal{E},\mathcal{E}'} \subseteq \Lambda^{\mathcal{E},\mathcal{E}'} \otimes_{\mathbb{Z}} \mathbb{R}$$

be the polyhedral cone generated by $[X] - [Y] + [Z]$, for all conflations $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E} \setminus \mathcal{E}'$.

Proposition

$\mathcal{C}^{\mathcal{E},\mathcal{E}'}$ is simplicial. Its extremal rays are given by AR-conflations in $\mathcal{E} \setminus \mathcal{E}'$. Its face lattice is isomorphic to the interval $[\mathcal{E}', \mathcal{E}]$.

Cones II

For a pair of exact structures $\mathcal{E}' < \mathcal{E}$, we define the (Hall algebra) *degree cone*:

$$\mathcal{D}^{\mathcal{E},\mathcal{E}'} := \{\mathbf{d} \in \mathbb{R}^{\mathrm{Ind}(\mathcal{A})} \mid \mathbf{d} \text{ induces an algebra filtration,} \mathrm{gr}_{\mathbf{d}}(\mathcal{H}(\mathcal{E})) = \mathcal{H}(\mathcal{E}')\}.$$

From Theorems I and II, we have:

$$\mathcal{D}^{\mathcal{E},\mathcal{E}'} = \{\varphi \in (K_0^{\mathrm{add}}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R})^* \mid \text{for any } x \in \mathcal{C}^{\mathcal{E},\mathcal{E}'}, \varphi(x) > 0; \text{ for any } y \in \mathcal{C}^{\mathcal{E}'},\ \varphi(y) = 0\}.$$

Up to linearity subspace, the cones $\mathcal{C}^{\mathcal{E},\mathcal{E}'}$ and $\mathcal{D}^{\mathcal{E},\mathcal{E}'}$ are polar dual to each other.

Comultiplication, quantum groups and Hall algebras

Theorem (Ringel-Green,...,Bridgeland, G., Lu-Peng,...)

Let Q be a finite acyclic quiver. Then

$$U_{\sqrt{q}}(\mathfrak{g}(Q)) \hookrightarrow \Big(\big(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\operatorname{mod} kQ), ab)/\mathcal{I}\big)\,[S^{-1}]\Big)_{red}.$$

This is an isomorphism if and only if Q is of Dynkin type.

$\mathcal{C}_{\mathbb{Z}/2}(\operatorname{mod} kQ)$ is the category of 2-periodic complexes:

$$M^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} M^1, \quad d^1 \circ d^0 = d^0 \circ d^1 = 0.$$

This is only an algebra map!

$$\operatorname{gldim}(\mathcal{C}_{\mathbb{Z}/2}(\operatorname{mod} kQ), \mathrm{ab}) = \infty.$$

So Green's comultiplication is not compatible with the multiplication.
Can we recover the comultiplication?

$(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})$ has Gorenstein dimension 1.

- $\text{gpr} = \mathcal{C}_{\mathbb{Z}/2}(\text{proj}\, kQ)$;
- $\text{gin} = \mathcal{C}_{\mathbb{Z}/2}(\text{inj}\, kQ)$;
- $\text{gpr}^{\perp} = {}^{\perp}\text{gin} = \mathcal{C}_{\mathbb{Z}/2,ac}(\text{mod}\, kQ)$;
- $\underline{\text{gpr}} \xrightarrow{\sim} \mathcal{D}_{\mathbb{Z}/2}(\text{mod}\, kQ)$.

Define an exact structure $\mathcal{E}_{CE}$ on $\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ)$ as follows:

$$A^{\bullet} \to B^{\bullet} \to C^{\bullet}$$

is a conflation if

$$A^i \to B^i \to C^i \quad \text{and} \quad H^i(A^{\bullet}) \to H^i(B^{\bullet}) \to H^i(B^{\bullet})$$

are short exact for $i = 0, 1$.

gpr become projectives and gin become injectives in $(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \mathcal{E}_{CE})$.

Theorem

$$U_{\sqrt{q}}(\mathfrak{g}(Q)) \hookrightarrow \left(\left(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \mathcal{E}_{CE})/\mathcal{I}\right)[S^{-1}]\right)_{red}$$

is a coalgebra homomorphism.

- $(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \mathcal{E}_{CE})$ is **hereditary.** But Green's theorem used the abelian exact structure, so it doesn't apply :(
- The RHS is a twisted extended Hall algebra of $(\text{gr}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})$. This category is **hereditary and abelian**!
- This induces a comultiplication on the RHS compatible with the multiplication. It coincides with Green's comultiplication w.r.t. $\mathcal{E}$.
- The RHS is an **algebra degeneration** of $\left(\left(\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})/\mathcal{I}\right)[S^{-1}]\right)_{red}$.

The comultiplication above is compatible with the multiplication of $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \text{ab})$.

Generalized quantum doubles

Conjecture

- $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), ab)$ *(before taking the quotient) is a quantum quasi-symmetric algebra in the sense of Fang-Rosso. It is the double of the Drinfeld double of* $\mathcal{H}_{tw}^{e}(\mathcal{A})$*, as considered by Joseph. This is a Hopf algebra, but the comultiplication is not the Hall multiplication of* $\mathcal{H}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), ab)$*.*
- $\mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\text{mod}\, kQ), \mathcal{E}_{CE})$ *realizes the tensor double of the Drinfeld double. The comultiplication is the Hall multiplication.*
- *All generalized quantum doubles are realized by Hall algebras of some of exact structures in* $[\mathcal{E}_{CE}, ab]$*.*
- *The quotient of* $\mathcal{H}_{tw}^{e}(\mathcal{A})$ *by the ideal* $\mathcal{I}$ *can be understood as a"relative integration map" along the classes of acyclic complexes. It is compatible with the change of an exact structure. It induces generalized quantum doubles of* $\mathcal{H}_{tw}^{e}(\mathcal{A})$*.*

Further directions

- Prove Theorem II in general case without using Auslander-Reiten theory. We have a conjectural approach, but it's too early to say anything.
- Non-additive case: proto-exact categories (Dyckerhoff-Kapranov).
- Cohomological HA, K-theoretic HA,... The PBW theorem is known for them, but it is proved differently.
- Degenerations of derived Hall algebras of triangulated categories (defined by Toën and Xiao-Xu).

Q1: What should replace ”substructures” in the setting of triangulated categories?
Q2: What structures do extension-closed subcategories of triangulated categories admit?

Extriangulated categories I

[Nakaoka-Palu, 2016]: Unify exact and triangulated categories. Axiomatize extension-closed subcategories of triangulated categories.
[Hu-Zhang-Zhou, 2019]: All “closed” substructures (“proper classes of triangles” of Beligiannis) of triangulated structures are extriangulated.
[Nakaoka-Palu, 2020]: Homotopy categories of exact (additive) $\infty-$categories carry natural extriangulated structures.

The class of extriangulated structures is closed under the following operations:

- Taking an extension-closed subcagegory;
- Taking a closed additive sub-bifunctor;
  Equivalently, taking a proper class of “conflations”;
- Taking a localization with respect to an *admissible model structure*; equivalently, w.r.t a *Hovey twin cotorsion pair*;
- Taking an ideal quotient by an ideal generated by morphisms $I \to P$ (from injectives to projectives).

Extriangulated structures and Hall algebras

(F.-G., in preparation)

- Define Hall-type algebras of extriangulated categories (with certain finiteness conditions) and prove their associativity. This recovers usual and derived Hall algebras.
- Generalize Theorems I and II to Hall algebras of extriangulated structures.

For the proofs we use [Iyama-Nakaoka-Palu], [Ogawa], [Enomoto],...

$\dim \mathrm{End}(-)$ is again a quasi-valuation w.r.t any extriangulated structure.

Corollary (generalizing Berenstein-Greenstein)

The PBW theorem holds for Hall-type algebras of extriangulated categories.

Hereditary case

[G.-Nakaoka-Palu, in preparation]: Study higher (positive and negative) extensions. Can define *hereditary* extriangulated categories $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

For those, the structure constants

$$u_A * u_C = \sum_{B \in \mathrm{Iso}(\mathcal{C})} \frac{|\mathbb{E}(A,C)_B|}{|\mathbb{E}(A,C)|} u_B$$

define an associative unital algebra.

Example

$\mathcal{E}_{CE}$ is a "lift" of a hereditary extriangulated structure on $\mathcal{D}_{\mathbb{Z}/2}(\text{mod } kQ)$, where a triangle is "proper" iff it is isomorphic to one given by an $\mathcal{E}_{CE}-$conflation.
This structure is neither exact, nor triangulated (but $\mathcal{D}_{\mathbb{Z}/2}(\text{mod } kQ)$ admits a triangulated structure!).

Extriangulated categories III

[Padrol-Palu-Pilaud-Plamondon, 2019] consider 2 concrete classes of extriangulated categories:

- Cluster categories with certain relative extriangulated structures.
- $K^{[-1,0]}(\operatorname{proj} \Lambda)$.

These structures are hereditary. In both cases, one can define $g-$vectors. In the additively finite case, they form $g-$vector fans and [PPPP] show that polytopal realizations of these fans are given by points in *type cones*.

Observation

Type cones of $g-$vector fans coincide with Hall algebra degree cones of these extriangulated categories!

F.-G.-Palu-Plamondon-Pressland, in progress: Explain this from the Hall algebra perspective and apply to (quantum) cluster algebras.