Linear quasi-categories as templicial modules

Joint work with Wendy Lowen

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Introduction

- Many models for $(\infty, 1)$-categories:
  Quasi-categories, simplicial categories, complete Segal spaces, ...
- Models for enriched $(\infty, 1)$-categories: Gepner-Haugseng [3], Lurie [8]
- We propose a different model for linear quasi-categories [7].

Intuitive approach:

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Let $(\mathcal{V}, \otimes, I)$ be a cocomplete monoidal closed category.

$\dim$	underlying graph	category-like structure
1	$\mathcal{V}$-enriched quiver	$\mathcal{V}$-enriched category
∞	templicial $\mathcal{V}$-object	$\mathcal{V}$-quasi-category

We will focus on the case $(\mathcal{V}, \otimes, I) = (\mathrm{Mod}(k), \otimes_k, k)$ for some fixed unital commutative ring k.

Overview

Simplicial sets

Definition

Let $\mathbf{\Delta}$ be the category of all posets $[n] = \{0, ..., n\}$ with $n \geq 0$ and order morphisms $f : [n] \to [m]$ between them.

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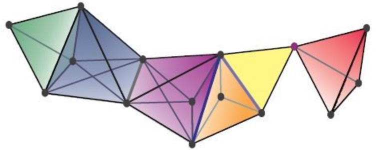

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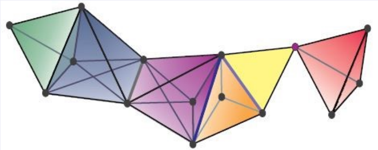


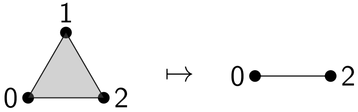
$\forall n \geq 0 : X_n = X([n])$ is the set of n-*simplices* of X.
We denote the category of simplicial sets by

$$\mathrm{SSet} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Set})$$

Equivalently, a simplicial set X is a family of sets $(X_n)_{n\geq 0}$ with for all $0 \leq i \leq n$:

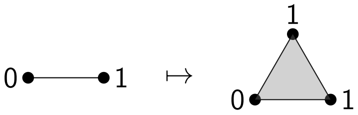
face maps:

$$d_i : X_n \to X_{n-1}$$



degeneracy maps:

$$s_i : X_n \to X_{n+1}$$



satisfying some identities.

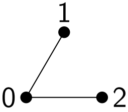
Standard simplices and horns

Definition

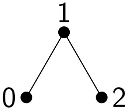
Let $n \geq 0$. The standard n-simplex is the simplicial set

$$\Delta^n = \mathbf{\Delta}(-, [n]) : \mathbf{\Delta}^{op} \to \mathrm{Set}$$

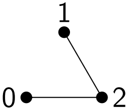
For $0 \leq j \leq n$, the jth horn Λ^n_j is obtained by removing the jth face and the interior from Δ^n.



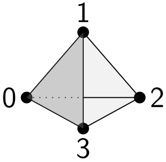
Λ^2_0



Λ^2_1



Λ^2_2

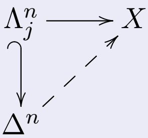


Λ^3_0

The Kan condition

Definition (Boardman-Vogt [1], Joyal [4])

A simplicial set X is called a quasi-category if it satisfies the weak Kan condition, that is for all $0 < j < n$, every diagram in SSet



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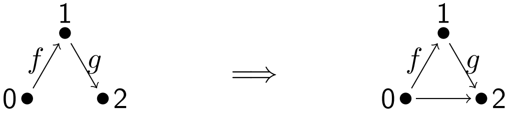
$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a lift. We denote the category of quasi-categories by QCat.
If such diagrams also lifts for $j = 0$ and $j = n$, we call X a Kan complex or ∞-groupoid.

E.g. for $j = 1, n = 2$:



This defines a composition on the edges of X which is only associative and unital up to coherent homotopy!
Slogan:

$$(\infty, 1)\text{-category theory} = \text{Category theory} + \text{Homotopy theory}$$

Examples

1. Let X be a topological space, then there is a ∞-groupoid $\mathrm{Sing}(X)$ which "contains all homotopic information of X".

0-simplices	1-simplices	2-simplices	...
points of X	paths in X	path homotopies	...

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2. Let $\mathcal{C}_\bullet$ be a dg-category, then there is a quasi-category $N^{dg}(\mathcal{C}_\bullet)$ called the dg-nerve of $\mathcal{C}_\bullet$ (Lurie [8]).

0-simplices	1-simplices	2-simplices	...
objects A of $\mathcal{C}_\bullet$	0-cycles $f \in \mathcal{C}_0(A,B)$	homotopies h $\partial(h) = f_{02} - f_{12}f_{01}$	...

Another example: Nerve of a category

Definition

Let $\mathcal{C}$ be a small category. Then its nerve is the simplicial set $N(\mathcal{C})$:

$$N(\mathcal{C})_n = \coprod_{A_0,\dots,A_n\in\mathcal{C}} \mathcal{C}(A_0,A_1)\times \dots \times \mathcal{C}(A_{n-1},A_n) \quad (\forall n \geq 0)$$

and for all $0 \leq i \leq n$,

$$d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1} :$$

$$(f_1,\dots,f_n) \mapsto \begin{cases} (f_2,\dots,f_n) & \text{if } i=0 \\ (f_1,\dots,f_{i+1}f_i,\dots,f_n) & \text{if } 0<i<n \\ (f_1,\dots,f_{n-1}) & \text{if } i=n \end{cases}$$

$$s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1} : (f_1,\dots,f_n) \mapsto (f_1,\dots,f_i,\mathrm{id},f_{i+1},\dots,f_n)$$

This extends to a fully faithful functor $N : \mathrm{Cat} \to \mathrm{QCat}$.

Nerve of a $\mathcal{V}$-enriched category?

Definition

Let $\mathcal{C}$ be a $\mathcal{V}$-category. Define, for all $n \geq 0$ and $A, B \in \mathcal{C}$:

$$N_{\mathcal{V}}(\mathcal{C})_n(A,B) = \coprod_{A_1,\dots,A_{n-1}\in\mathcal{C}} \mathcal{C}(A,A_1) \otimes \dots \otimes \mathcal{C}(A_{n-1},B)$$

and for all $0 \leq i \leq n$,

$$d_i : N_{\mathcal{V}}(\mathcal{C})_n(A,B) \to N_{\mathcal{V}}(\mathcal{C})_{n-1}(A,B)$$
$$= \begin{cases} ? & \text{if } i = 0 \\ \mathrm{id}_{\mathcal{C}(A,A_1)} \otimes \dots \otimes m_{A_{i-1},A_i,A_{i+1}} \otimes \dots \otimes \mathrm{id}_{\mathcal{C}(A_{n-1},B)} & \text{if } 0 < i < n \\ ? & \text{if } i = n \end{cases}$$

$$s_i : N_{\mathcal{V}}(\mathcal{C})_n(A,B) \to N_{\mathcal{V}}(\mathcal{C})_{n+1}(A,B) :$$
$$= \mathrm{id}_{\mathcal{C}(A,A_1)} \otimes \dots \otimes u_{A_i} \otimes \dots \otimes \mathrm{id}_{\mathcal{C}(A_{n-1},B)}$$

$N_{\mathcal{V}}(\mathcal{C})$ is not a simplicial object!

Definition

Let $\boldsymbol{\Delta}_{\mathrm{f}}$ be the subcategory of $\boldsymbol{\Delta}$ of all morphisms $f : [m] \to [n]$ such that $f(0) = 0$ and $f(m) = n$.

This is a monoidal category with $[m] + [n] = [m + n]$.

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$N_{\mathcal{V}}(\mathcal{C})$ is not a simplicial object, but the data above can be organized into a strong monoidal functor

$$N_{\mathcal{V}}(\mathcal{C}) : {\mathbf{\Delta}_{\mathrm{f}}}^{op} \to \mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}(\mathcal{V})$$

In fact, any strong monoidal functor ${\mathbf{\Delta}_{\mathrm{f}}}^{op} \to \mathrm{Quiv}_{\mathrm{Ob}(\mathcal{C})}(\mathcal{V})$ is of the form $N_{\mathcal{V}}(\mathcal{C})$ for some $\mathcal{V}$-enriched category $\mathcal{C}$.

Templicial objects

Definition

A tensor-simplicial object or templicial object of $\mathcal{V}$ is a pair (X, S) with S a set and

$$X : \mathbf{\Delta}_{\mathrm{f}}{}^{op} \to \mathrm{Quiv}_S(\mathcal{V})$$

a strongly unital, colax monoidal functor, with comultiplication

$$(\mu_{p,q} : X_{p+q} \to X_p \otimes_S X_q)_{p,q \geq 0}$$

and counit $\epsilon : X_0 \xrightarrow{\sim} I_S$.
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We denote the category of templicial objects by $\mathrm{S}_{\otimes}\,\mathcal{V}$.

We should view S as the set of vertices of X.
For all $n \geq 0$ and $a, b \in S$, we should view $X_n(a, b) \in \mathcal{V}$ as the object of n-simplices with first vertex a and last vertex b.

Proposition (Leinster [6])

If $\mathcal{V} = \mathrm{Set}$*, then* $\mathrm{S}_{\otimes}\,\mathcal{V} \simeq \mathrm{SSet}$*.*

Proposition

There is a fully faithful functor $N_{\mathcal{V}} : \mathrm{Cat}(\mathcal{V}) \to \mathrm{S}_{\otimes}\,\mathcal{V}$

Proposition

The category $\mathrm{S}_{\otimes}\,\mathcal{V}$ *is cocomplete and thus there is an adjunction*

$$\tilde{F} : \mathrm{SSet} \leftrightarrows \mathrm{S}_{\otimes}\,\mathcal{V} : \tilde{U}$$

Linear quasi-categories

From now on, we restrict to the case $\mathcal{V} = \mathrm{Mod}(k)$.

Definition

A templicial module X is a k-linear quasi-category if it satisfies the weak Kan condition, that is for all $0 < j < n$, every diagram in $\mathrm{S}_{\otimes}\,\mathrm{Mod}(k)$

$$\begin{array}{ccc} \tilde{F}(\Lambda^n_j) & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ \tilde{F}(\Delta^n) & & \end{array}$$

has a lift. We denote the category of k-linear quasi-categories by $\mathrm{QCat}(k)$.

Proposition

A templicial module X is a k-linear quasi-category if and only if $\tilde{U}(X)$ is a quasi-category.

Theorem

There is a diagram of adjunctions

$$\begin{array}{ccc} \mathrm{Cat} & \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} & \mathrm{Cat}(k) \\ h \uparrow\downarrow N & & h_k \uparrow\downarrow N_k \\ \mathrm{QCat} & \underset{\tilde{U}}{\overset{\tilde{F}}{\rightleftarrows}} & \mathrm{QCat}(k) \end{array}$$

which commutes in the following sense:

$$N_k \circ \mathcal{F} \simeq \tilde{F} \circ N \qquad \tilde{U} \circ N_k \simeq N \circ \mathcal{U}$$

$$\mathcal{F} \circ h \simeq h_k \circ \tilde{F} \qquad h \circ \tilde{U} \simeq \mathcal{U} \circ h_k$$

Definition

Let k be a field. A Frobenius algebra is a finite-dimensional k-algebra A equipped with a k-linear map $\epsilon : A \to k$ such that $\ker(\epsilon)$ contains no non-trivial left ideal of A.

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Examples

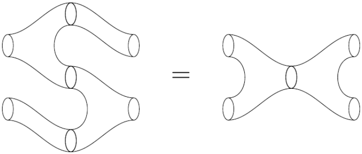
1. Matrix algebra $M_n(k)$ with $\epsilon = tr$ the trace map.
2. Group algebra $k[G]$ for a finite group G with ϵ projecting onto $k1_G$.
3. Any zero-dimensional local Gorenstein ring which is finite dimensional over its residu field is a Frobenius algebra.

Proposition (Lawvere [5])

A Frobenius algebra is equivalent to a k-algebra A equipped with a coalgebra structure

$$\mu : A \to A \otimes A \quad \textit{and} \quad \epsilon : A \to k$$

such that $\mu \circ m = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \mu) = (\mathrm{id}_A \otimes m)(\mu \otimes \mathrm{id}_A)$.



$$cFrob_k \simeq SymMonFun(2Cob, \mathrm{Vect}(k)) = 2TQFT_k$$

We can generalize this to:

Definition (Day-Pastro [2])

A Frobenius monoidal functor between monoidal categories $\mathcal{U}$ and $\mathcal{V}$ is a functor $F : \mathcal{U} \to \mathcal{V}$ with a lax structure (m, u) and a colax structure (μ, ϵ) such that for all $A, B, C \in \mathcal{U}$:

$$\begin{aligned} \mu_{A\otimes B,C} \circ m_{A,B\otimes C} &= (m_{A,B} \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \mu_{B,C}) \\ \mu_{A,B\otimes C} \circ m_{A\otimes B,C} &= (\mathrm{id}_A \otimes m_{B,C})(\mu_{A,B} \otimes \mathrm{id}_C) \end{aligned} \tag{1}$$

Example

A Frobenius monoidal functor $* \to \mathrm{Vect}(k)$ is a Frobenius k-algebra.

Example

The tensor algebra $T(V[1])$ of an $\mathbb{N}$-graded vectorspace $V_\bullet$ is Frobenius monoidal when considered as a functor $\mathbb{N} \to \mathrm{Vect}(k)$.

Frobenius templicial modules

Definition

Let X be a templicial module with colax structure (μ, ϵ).
A nonassociative Frobenius (naF) structure on X is a *non-associative* lax structure (m, u) on X such that μ and m satisfy (1) and $u = \epsilon^{-1}$.

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Proposition

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2. *Let X be templicial module with a naF-structure. Then X is a linear quasi-category.*

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Corollary

The functor $\tilde{F}$ sends quasi-categories to linear quasi-categories.

Relation with dg-categories

Denote the category of templicial modules with *associative* Frobenius structures by

$$S_{\otimes}^{Frob}\,\mathrm{Mod}(k)$$

Theorem

There is an equivalence of categories

$$dg\,\mathrm{Cat}_{\geq 0}(k) \leftrightarrows S_{\otimes}^{Frob}\,\mathrm{Mod}(k)$$

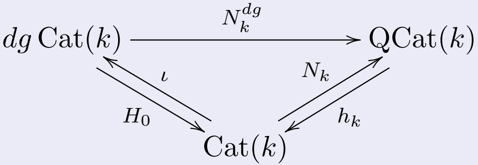
This equivalence induces a functor

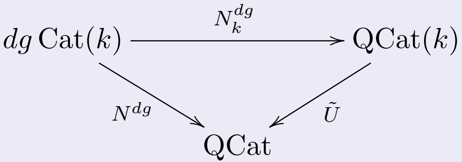
$$N_k^{dg} : dg\,\mathrm{Cat}(k) \to \mathrm{QCat}(k)$$

called the linear dg-nerve.

Proposition

The following diagrams of functors commute up to natural isomorphism:





Thanks for listening!

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