Categorification of representation theory with an application to Soergel bimodules

(joint work with Mackaay, Mazorchuk, Tubbenhauer and Zhang)

Vanessa Miemietz
(University of East Anglia)

Motivation

Why 2-representation theory?

- monoidal categories and 2-categories becoming more and more important, leading to advances in representation theory
  - homological data on group- and Lie-theoretic objects
  - proof of Broué's abelian defect group conjecture for symmetric groups
  - algebraic proof of Kazhdan-Lusztig conjectures for all Coxeter types via Soergel bimodules
  - counterexamples to James' conjectures for symmetric groups
- study 2-categories using 2-representation theory

Note: All categories in this talk are assumed to be locally small (or small if necessary). Further, $\Bbbk$ is an algebraically closed field.

Some classical results in representation theory

Let A be an algebra. Consider $A/\mathrm{rad}(A) \cong \prod_{i=1}^{r} M_{n_i}(\Bbbk)$.

We can assume $1_A = \sum_{i=1}^{r} 1_{M_{n_i}(\Bbbk)}$, and each $1_{M_{n_i}(\Bbbk)} = e_{i1} + \cdots + e_{in_i}$ can be decomposed into primitive orthogonal idempotents.

$\exists$ bijection $\{\text{simple } A\text{-modules}\}/\cong \leftrightarrow \{e_{i1}Ae_{i1}/\mathrm{rad}(e_{i1}Ae_{i1}) | i = 1, \ldots, r\}$.

Moreover, for a simple A-module S, we have a double centraliser theorem:

$$\mathrm{End}_{\mathrm{End}_A(S)}(S) \cong A/\mathrm{ann}(S)$$

If S is the simple module labelled by i, then $A/\mathrm{ann}(S) \cong M_{n_i}(\Bbbk)$ is Morita equivalent to $\mathrm{End}_A(S) \cong \Bbbk$.

Upshot: All of this categorifies in some form, but not on the the nose (no semisimplicity modulo radical, each 'matrix ring' analogue can have lots of simples, etc).

2-categories

Definition. A 2-**category** $\mathscr{C}$ consists of

- a class (or set) $\mathscr{C}$ of objects;
- for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms from $\mathtt{i}$ to $\mathtt{j}$
  - objects in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 1-**morphisms** of $\mathscr{C}$,
  - morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 2-**morphisms** of $\mathscr{C}$;
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity 1-morphisms $\mathbb{1}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathscr{C}$;
- natural (strict) axioms.

Weak axioms yield a **bicategory**.

General examples of 2-categories

Examples.

- A (strict) monoidal category $\mathcal{C}$ is a 2-category with one object, which has the objects of $\mathcal{C}$ as 1-morphisms, and the morphisms of $\mathcal{C}$ as 2-morphisms.

- the 2-category **Cat** of small categories (1-morphisms are functors and 2-morphisms are natural transformations);

- the 2-category $\mathfrak{A}_{\Bbbk}^{f}$ whose
  - objects are small idempotent complete $\Bbbk$-linear categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces (that is, equivalent to the category of finitely generated projective modules over a finite-dimensional $\Bbbk$-algebra);
  - 1-morphisms are additive $\Bbbk$-linear functors;
  - 2-morphisms are natural transformations.

Finitary 2-categories

Definition. A 2-category $\mathscr{C}$ is **finitary** over $\Bbbk$ if

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i},\mathtt{j})$ is in $\mathfrak{A}_{\Bbbk}^{f}$ (i.e. equivalent to A-proj for some algebra A);
- composition is biadditive and $\Bbbk$-bilinear;
- identity 1-morphisms are indecomposable.

Moral: Finitary 2-categories are 2-analogues of finite dimensional algebras.

Definition. A 2-category $\mathscr{C}$ is **fiat** (**f**initary - **i**nvolution - **a**djunction - **t**wo category) if

- it is finitary;
- there is a weak involutive equivalence $(-)^{*}\colon \mathscr{C} \to \mathscr{C}^{\mathrm{op,op}}$ such that there exist adjunction morphisms $F \circ F^{*} \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^{*} \circ F$.

Examples

- tensor categories (only weakly fiat)
- fusion categories (semi-simple tensor categories)
- projective endofunctors of $A\text{-}\mathrm{mod}$ (finitary for finite dimensional A, fiat if A weakly symmetric)
- finitary quotients of Kac–Moody 2-categories (aka KLR 2-categories, categorified quantum groups)
- Soergel bimodules (aka Hecke 2-categories)

From now on, $\mathscr{C}$ denotes a fiat 2-category.

2-representations

Definition. A **finitary** 2-**representation M** of $\mathscr{C}$ is a (strict) 2-functor $\mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$, i.e.

- $\mathbf{M}(\mathtt{i}) \approx B_{\mathtt{i}}\text{-proj}$ for some algebra $B_{\mathtt{i}}$;
- for $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$, $\mathbf{M}(\mathrm{F})\colon \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$ is an additive functor;
- for $\alpha\colon \mathrm{F} \to \mathrm{G}$, $\mathbf{M}(\alpha)\colon \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ is a natural transformation.

Example.

- For each object $\mathtt{i}$ in $\mathscr{C}$, we have the **principal** 2-representation $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.
- Projective A-A-bimodules acting on $A\text{-proj}$.

Definition. **M** is **simple transitive** if $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}$-stable ideals.

Goal. Classify simple transitive 2-representations for interesting 2-categories.

Cell combinatorics for 2-categories

$\Sigma(\mathscr{C})$, the set of isoclasses of indecomposable 1-morphisms in $\mathscr{C}$, has several partial preorders.

left preorder: $\mathrm{F} \geq_L \mathrm{G}$ if $\exists \mathrm{H}$ such that F is a direct summand of HG

left cells: equivalence classes w.r.t. $\geq_L$

Similarly:

right preorder: $\mathrm{F} \geq_R \mathrm{G}$ if $\exists \mathrm{H}$ such that F is a direct summand of GH

right cells: equivalence classes w.r.t. $\geq_R$

two-sided preorder: $\mathrm{F} \geq_J \mathrm{G}$ if $\exists \mathrm{H}_1, \mathrm{H}_2$ such that F is a direct summand of $\mathrm{H}_1\mathrm{G}\mathrm{H}_2$

two-sided cells: equivalence classes w.r.t. $\geq_J$

Cell combinatorics for 2-categories

Example. Cells for the 2-category $\mathscr{S}$ of Soergel bimodules are Kazhdan–Lusztig cells.

E.g. let $W = \langle s, t | s^2 = 1 = t^2, stst = tsts \rangle$ of type B_2. Cells are given by

1

s, sts	st
ts	t, tst

$stst$

An $\mathcal{H}$-**cell** is the intersection of a left and a right cell.

A two-sided cell is **strongly regular**, if every $\mathcal{H}$-cell in it has precisely one element.

Simple transitive 2-representations

Lemma. [Chan–Mazorchuk] Every simple transitive 2-representation $\mathbf{M}$ has an **apex**, which is the unique maximal two-sided cell $\mathcal{J}$ such that $\mathbf{M}(\mathcal{J}) \neq 0$.

$\rightsquigarrow$ study simple transitive 2-representations apex by apex

To each left cell $\mathcal{L}$ in $\mathscr{C}$, we can associate a **cell** 2-representation $\mathbf{C}_{\mathcal{L}}$, which is simple transitive by construction.

Theorem. [Mazorchuk–M.] If the apex of a simple transitive 2-representation $\mathbf{M}$ is strongly regular, $\mathbf{M}$ is equivalent to a cell 2-representation.

Known: Simple transitive implies cell for

- appropriate quotients of Kac–Moody 2-categories [Mazorchuk–M, Macpherson];
- 2-categories of projective bimodules [Mazorchuk–M, Mazorchuk–M–Zhang];
- Soergel bimodules in type A, but **not** in other types.

$\mathcal{H}$-cell reduction

Let $\mathcal{L} \subseteq \mathcal{J}$ be a left cell in $\mathscr{C}$ and set $\mathcal{H} = \mathcal{L} \cap \mathcal{L}^*$.

Construct $\mathscr{C}_{\mathcal{H}}$ in several steps:

- take quotients by all two-sided cells $\mathcal{J}' \nleq \mathcal{J}$;
- inside quotient, take additive closure of $\mathbb{1}_{\mathtt{i}(\mathcal{H})}$ and 1-morphisms in $\mathcal{H}$;
- factor out the maximal ideal not containing id_{F} for $\mathrm{F} \in \mathcal{H}$.

Theorem. [Mackaay–Mazorchuk–M–Zhang] There is a bijection

$$\{\text{simple transitive 2-representations of } \mathscr{C} \text{ with apex } \mathcal{J}\}$$
$$\updownarrow$$
$$\{\text{simple transitive 2-representations of } \mathscr{C}_{\mathcal{H}} \text{ with apex } \mathcal{H}\}$$

Upshot: concentrate on $\mathscr{C}_{\mathcal{H}} \rightsquigarrow$ smaller! We call this **$\mathcal{H}$-cell reduction**.

Double Centraliser Theorem

Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_{\mathcal{H}}$.

There is a canonical 2-functor

$$\mathrm{can}\colon \mathscr{C}_{\mathcal{H}} \to \mathscr{E}nd_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}).$$

Theorem. *[Double Centraliser Theorem]*
There is an equivalence of 2-semicategories

$$\mathscr{E}nd^{inj}_{\mathscr{E}nd_{\mathscr{C}_{\mathcal{H}}}(\mathbf{M})}(\mathbf{M}) \simeq \mathrm{add}(\mathcal{H}),$$

where ^{inj} refers to restricting to injective endofunctors.

Hecke algebras

(W, S) Coxeter group

$W = \langle s_i |, s_i \in S, s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$ for some $m_{ij} \in \{2, 3, \cdots, \infty\}$

The **Hecke algebra** $\mathrm{H}(W)$ associated to W is a quantisation of $\mathbb{Z}W$, and has an associated **cell theory** (Kazhdan–Lusztig cells). To a **two-sided cell** J and each intersection H of a **left cell** with its associated **right cell**, Lusztig associates an **asymptotic Hecke algebra**, and there is a bijection

$$\{\text{simple representations of the asymptotic algebras}\}$$
$$\updownarrow$$
$$\{\text{simple representations of the Hecke algebra}\}$$

Idea: Asymptotic algebras are easier to understand. They are essentially matrix algebras with a modified multiplication, but the classification of simple module is not affected.

Soergel bimodules or the Hecke 2-category

(W, S, V) finite Coxeter system, V reflection representation

$R = \mathbb{C}[V]/(\mathbb{C}[V]^W)_+$ coinvariant algebra

$R_i := R \otimes_{R^{s_i}} R$ for $s_i \in S$

The 2-**category** $\mathscr{S} = \mathscr{S}_{W,S,V}$ **of Soergel bimodules** or **Hecke** 2-**category** has

- one object $\varnothing$ (identified with R-proj);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with direct summands of direct sums of finite tensor products (over R) of the R_i;
- 2-morphisms are all natural transformations (bimodule morphisms).

Fact: Indecomposable 1-morphisms are labelled by elements in W, and $\mathscr{S}$ categorifies the Hecke algebra. In particular, indecomposable 1-morphism descend to a cellular basis (the KL-basis). [Soergel]

Classification of simple transitive 2-representations?

$\mathcal{H}$-cell reduction reduces classification of simple transitive 2-representations of $\mathscr{S}$ to $\mathscr{S}_{\mathcal{H}}$, where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in each two-sided cell.

Let $\mathbf{C}_{\mathcal{H}}$ be the cell 2-representation of $\mathscr{S}_{\mathcal{H}}$ associated to $\mathcal{H}$. The double centraliser theorem specialises to an equivalence of 2-semicategories

$$\mathscr{E}nd^{inj}_{\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}})}(\mathbf{C}_{\mathcal{H}}) \simeq \mathrm{add}(\mathcal{H}).$$

To $\mathscr{S}_{\mathcal{H}}$, associate the **asymptotic bicategory** $\mathscr{A}_{\mathcal{H}}$. This categorifies Lusztig's asymptotic Hecke algebra. [Lusztig, Elias-Williamson]

$\mathscr{A}_{\mathcal{H}}$ is fusion (i.e. semisimple) and for almost all $\mathcal{H}$-cells, $\mathscr{A}_{\mathcal{H}}$ is well-understood and its simple transitive 2-representations have been classified. [Ostrik et al.]

Classification of simple transitive 2-representations?

Theorem. There is a biequivalence

$$\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}_{\mathcal{H}}) \simeq \mathscr{A}_{\mathcal{H}}$$

Caution: Ignoring gradings here for nicer statements!

Using these results, we can show:

Theorem. There is an biequivalence of 2-categories

$$\{\text{(graded) simple transitive 2-representations of } \mathscr{A}_{\mathcal{H}}\}$$
$$\updownarrow$$
$$\{\text{(graded) simple transitive 2-representations of } \mathscr{S}_{\mathcal{H}} \text{ with apex } \mathcal{H}\}$$

Upshot. Reduces classification problem to a well-studied one.

Thank you!

Thank you for your attention!