Homological mirror symmetry for invertible polynomials in two variables

Matthew Habermann

October 8, 2020

Background

Let $A = (a_{ij})$ be an invertible $n \times n$ matrix with integer coefficients. To any such A, we can associate a polynomial

$$\mathbf{w} = \sum_{i=1}^{n} \prod_{j=1}^{n} x^{a_{ij}}.$$

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We can also associate a polynomial to A^T, called the *Berglund–Hübsch transpose*, defined as

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Let $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$. Then $\mathbf{w} = x^3 y + y^2$, and $\check{\mathbf{w}} = x^3 + y^2 x$.

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Definition 2

Let A, $\mathbf{w}$, and $\check{\mathbf{w}}$ be as above. If both $\mathbf{w}$ and $\check{\mathbf{w}}$ define isolated singularities at the origin, and are both quasi-homogeneous, then we say that $\mathbf{w}$ is invertible.

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Example 3

Let $A = \begin{pmatrix} \ell & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ be the matrix which yields $\mathbf{w} = x^\ell + xy^2 + z^2$, the $D_{\ell-1}$ singularity. This is the Thom–Sebastiani sum of a chain and Fermat polynomial.

Background

The *maximal symmetry group* is defined as:

$$\Gamma_{\mathbf{w}} := \{(t_1, \dots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} |\ \mathbf{w}(t_1 x_1, \dots, t_n x_n) = t_{n+1} \mathbf{w}(x_1, \dots, x_n)\}.$$

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Conjecture 1 (Takahashi '10, Ueda '06, Futaki–Ueda '11, Lekili-Ueda '18)

For any invertible polynomial $\mathbf{w}$*, there is a quasi-equivalence*

$$\mathcal{FS}(\check{\mathbf{w}}) \simeq \mathrm{mf}(\mathbb{A}^n, \Gamma_{\mathbf{w}}, \mathbf{w})$$

of pre-triangulated A_∞ *categories over* $\mathbb{C}$*.*

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Consider $\check{V}$ the Milnor fibre of $\check{\mathbf{w}}$, and

$$Z_w := [(\operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n]/(w + x_0 x_1 \ldots x_n) \setminus (0))/\Gamma_{\mathbf{w}}].$$

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Conjecture 2 (Lekili-Ueda '18)

For any invertible polynomial $\mathbf{w}$ *of log general type, there is a quasi-equivalence*

$$\mathcal{F}(\check{V}) \simeq \text{perf } Z_{\mathbf{w}},$$
$$\mathcal{W}(\check{V}) \simeq D^b(Coh\ Z_{\mathbf{w}})$$

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Background

For the rest of the talk, we will restrict ourselves to $n = 2$.

Theorem 1 (Smith – H. '19)

For $\mathbf{w}$ *an invertible polynomial in two variables, there is a quasi–equivalence*

$$\mathcal{FS}(\check{\mathbf{w}}) \simeq \mathrm{mf}(\mathbb{A}^2, \Gamma_{\mathbf{w}}, \mathbf{w})$$

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Theorem 2 (H. '20)

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where Y is the projectivised stack $[(\mathbf{w}^{-1}(0) \setminus \{0\})/\Gamma_{\mathbf{w}}]$, and $\mathcal{C}$ is a subcategory of certain graded shifts of the origin.

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- The $(p-1)(q-1)$ objects corresponding to the structure sheaf of the origin and its fattenings.

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This is given as the path algebra of the following quiver with relations:

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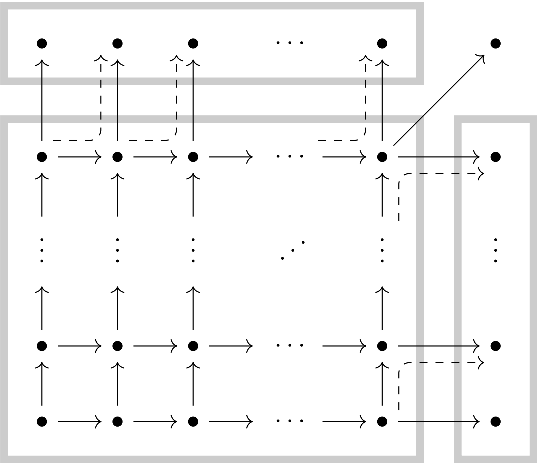

Relations:

(i) Squares commute

(ii) Dashed compositions vanish

Strategy of proof of Theorem 1.

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- Observe that End $\mathcal{E}$ is concentrated in degree 0 $\implies$ Intrinsically formal
- Therefore, split generates $\mathrm{mf}(\mathbb{A}^2, \Gamma_{\mathbf{w}}, \mathbf{w})$.
- Construct a Lefschetz fibration whose vanishing cycles match.

The A–side

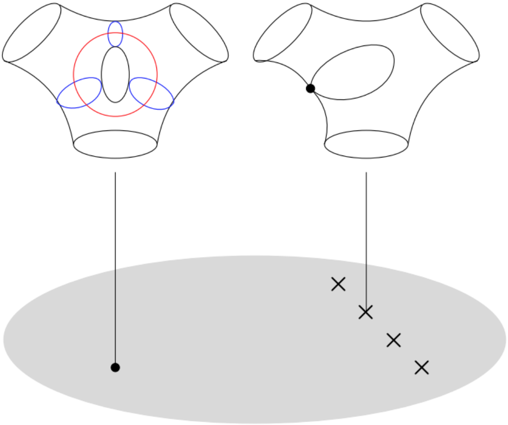

Figure: Lefschetz fibration corresponding to $x^2y + y^2x$

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There is a restriction functor

$$\begin{aligned} \mathcal{F}(\check{\mathbf{w}}) &\to \mathcal{F}(\check{V}), \\ \Delta_i &\mapsto \partial\Delta_i =: V_i, \end{aligned}$$

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$$A := \mathrm{End}(\bigoplus_{i=1}^{pq} V_i) = A^{\to} \oplus (A^{\to})^{\vee}[-1],$$

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This is the trivial extension algebra of degree 1 of $A^{\to}$.

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Implication

The subcategory of band modules of the corresponding gentle algebra has non-trivial higher products.

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Let A be a graded algebra, and consider the set of minimal A_∞-structures on the algebra.

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- Deduce the mirror by computable invariants.

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The *semi-universal unfoldings* of $\mathbf{w}$ are given by

$$\tilde{\mathbf{w}} = \mathbf{w} + \sum_{(i,j)\in J_{\mathbf{w}}} u_{ij} x^i y^j$$

with $U = \mathrm{Spec}\ \mathbb{C}[u_1, \dots, u_\mu]$ its parameter space.

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with $U = \operatorname{Spec} \mathbb{C}[u_1, \ldots, u_\mu]$ its parameter space. Therefore, $\tilde{\mathbf{w}}$ is a map

$$\tilde{\mathbf{w}} : \mathbb{C}^2 \times U \to \mathbb{C},$$

and we define

$$\mathbf{w}_u := \tilde{\mathbf{w}}|_{\mathbb{C}^2 \times \{u\}}.$$

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In this case, we have that $\deg x = 1$, $\deg y = 2 = \deg z$, and $\deg \mathbf{w} = 5$. Therefore $\mathbf{w}_{u_{2,1}} \notin U_+$, but $\mathbf{w}_{u_{1,1}} \in U_+$, since

$$\mathbf{W}_{u_{1,1}} = x^3y + y^2x + u_{1,1}xyz$$

is quasi-homogeneous of degree 5.

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Key point

This is independent of $u \in U_+$!

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Polynomial untwisted $\implies$ This is an isomorphism.

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The story so far

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- On the B-side, we have identified a family of curves such that:
  1. The A_∞ algebra of the object which split generates perf Y_u defines an A_∞-structure on A.
  2. *Every* A_∞-structure on A arises as the chain-level endomorphism algebra of a generating object of perf Y_u for some $u \in U_+$.

Identifying the mirror

Theorem 3 (Lekili–Ueda '18)

Let $\check{\mathbf{w}}$ be the transpose of an invertible polynomial such that $\check{V}$ is of log general type. Then

$$\mathrm{SH}^*(\check{V}) \simeq \mathrm{HH}^*(\mathcal{F}(\check{V})).$$

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Sketch of proof of Theorem 1.

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- Unless $u = u_{1,1}$, we have that rank $\mathrm{HH}^*(Y_u) <$ rank $\mathrm{SH}^*(\check{V})$.
- Since we know that there *must* be a $u \in U_+$ for which Y_u is mirror to $\check{V}$, the only possibility is $u = u_{1,1}$, and $Y_u = Z_w$.

□

The end

Thank you!