τ-Tilting Finite Algebras With Non Empty Left or Right Parts are Representation-Finite

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Introduction: τ-Tilting Theory

τ-tilting theory was introduced by Adachi, Iyama and Reiten [1] as a far-reaching generalization of classical tilting theory for finite dimensional associative algebras. One of the main classes of objects in the theory is that of τ-rigid modules: a module M over an algebra Λ is *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, such a module M is called *τ-tilting* if the number $|M|$ of non-isomorphic indecomposable summands of M equals the number of isomorphism classes of simple Λ-modules.

Introduction: τ-Tilting Finite Algebras

Recently, a new class of algebras were introduced by Demonet, Iyama, Jasso [10] called τ-*tilting finite* algebras. They are defined as finite dimensional algebras with only a finite number of isomorphism classes of basic τ-tilting modules.

Introduction: Significance

If Λ is a τ-tilting finite algebra:

- There are finitely many support τ-tilting modules. [10].
- Every torsion class is functorially finite [10].
- There are finitely many bricks in its module category [10].
- The representation theory is easier to understand.

Introduction: Goal

An obvious sufficient condition to be τ-tilting finite is to be representation-finite. In general, this condition is not necessary. The aim of this talk is to prove for algebras Λ such that $\mathcal{L}_\Lambda$ or $\mathcal{R}_\Lambda \neq \emptyset$, representation-finiteness and τ-tilting finiteness are equivalent conditions.

Theorem 1.

Let Λ be a finite dimensional algebra such that $\mathcal{L}_\Lambda$ or $\mathcal{R}_\Lambda \neq \emptyset$. Then Λ is τ-tilting finite if and only if Λ is representation-finite.

Introduction: Notation

- Λ is f. d. over an algebraically closed field k.
- $\operatorname{mod}\Lambda$.
- $\operatorname{ind}\Lambda$.
- $\operatorname{add} M$.
- $\operatorname{Gen} M$.
- $\operatorname{Cogen} M$.
- $\Gamma(\operatorname{mod}\Lambda)$.
- $\operatorname{pd}_\Lambda M$ and $\operatorname{id}_\Lambda M$.

Torsion Pairs: Definition

For a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ we let

$$\mathcal{C}^{\perp} := \{X \in \operatorname{mod}\Lambda | \operatorname{Hom}_{\Lambda}(\mathcal{C}, X) = 0\}.$$

Dually, we define ${}^{\perp}\mathcal{C}$. We say a full subcategory $\mathcal{T}$ of $\operatorname{mod}\Lambda$ is a *torsion class* (respectively *torsionfree class*) if it is closed under factor modules (respectively, submodules) and extensions. A pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. In this case $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class.

Torsion Pairs: Ext-Projectives and Ext-Injectives

We say $X \in \mathcal{T}$ is Ext-*projective* (respectively, Ext-*injective*) if $\mathrm{Ext}^1_\Lambda(X, \mathcal{T}) = 0$ (respectively, $\mathrm{Ext}^1_\Lambda(\mathcal{T}, X) = 0$). Denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable Ext-projective objects in $\mathcal{T}$ up to isomorphism. Similarly, denote by $I(\mathcal{F})$ the direct sum of one copy of each of the indecomposable Ext-injective objects in $\mathcal{F}$ up to isomorphism.

Torsion Pairs: Contravariantly and Covariantly Finite

We recall that a full additive subcategory $\mathcal{C}$ of mod Λ is called *contravariantly finite* if, for any Λ-module M, there exists a morphism $f_{\mathcal{C}} : M_{\mathcal{C}} \longrightarrow M$ such that $M_{\mathcal{C}} \in \mathcal{C}$ and, if $f : N \longrightarrow M$ is any morphism with $N \in \mathcal{C}$, then there exists $g : N \longrightarrow M_{\mathcal{C}}$ such that $f = f_{\mathcal{C}}g$. The dual notion is that of *covariantly finite*. If $\mathcal{C}$ is both contravariantly and covariantly finite, then $\mathcal{C}$ is *functorially finite*.

Torsion Pairs: Functorially Finite

Proposition 2 ([9], [11], [13]).

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod}\Lambda$. Then the following are equivalent:

(a) *$\mathcal{T}$ is functorially finite.*

(b) *$\mathcal{F}$ is functorially finite.*

(c) *$\mathcal{T} = \operatorname{Gen} P(\mathcal{T})$.*

(d) *$\mathcal{F} = \operatorname{Cogen} I(\mathcal{F})$.*

Left Supported Algebras: Left and Right Parts (1)

Given $X, Y \in \operatorname{ind}\Lambda$, we denote $X \rightsquigarrow Y$ in case there exists a chain of nonzero nonisomorphisms

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots X_{t-1} \xrightarrow{f_t} X_t = Y$$

with $t \geq 0$, between indecomposable modules. In this case we say X is a predecessor of Y and Y is a successor of X. If $Y = X$, we say X lies on a cycle. We now recall the definition of the left and right part of a module category.

Left Supported Algebras: Left and Right Parts (2)

We denote by $\mathcal{L}_\Lambda$ and $\mathcal{R}_\Lambda$ the following subcategories of $\operatorname{ind}\Lambda$:

$$\mathcal{L}_\Lambda = \{Y \in \operatorname{ind}\Lambda : \operatorname{pd}_\Lambda X \leq 1 \text{ for each } X \rightsquigarrow Y\}.$$

$$\mathcal{R}_\Lambda = \{Y \in \operatorname{ind}\Lambda : \operatorname{id}_\Lambda X \leq 1 \text{ for each } Y \rightsquigarrow X\}$$

We call $\mathcal{L}_\Lambda$ the *left part* of the module category $\operatorname{mod}\Lambda$ and $\mathcal{R}_\Lambda$ the *right part*. It is easy to see that $\mathcal{L}_\Lambda$ is closed under predecessors while $\mathcal{R}_\Lambda$ is closed under successors.

Left Supported Algebras: Definition

An algebra Λ is called *left supported* provided the class add $\mathcal{L}_\Lambda$ is contravariantly finite in mod Λ. We define dually *right supported algebras*.

Left Supported Algebras: Characterizations

Let E be the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-injectives in add $\mathcal{L}_\Lambda$. Let F be the direct sum of a complete set of representatives of the isomorphism classes of indecomposable projectives not lying in $\mathcal{L}_\Lambda$.

Theorem 3.

[5, *Theorem A*] *Let Λ be an algebra. The following are equivalent:*

- (a) *Λ is left supported.*
- (b) *add $\mathcal{L}_\Lambda = CogenE$.*
- (c) *$T = E \oplus F$ is a tilting module.*

Left Supported Algebras: Components (1)

We need a result on the structure of the Auslander-Reiten components of a left supported algebra Λ. We recall a connected component Γ of $\Gamma(\text{mod}\,\Lambda)$ is called a *postprojective component* if Γ does not contain an oriented cycle and each indecomposable module $X \in \Gamma$ is of the form $\tau^{-r}P$ for some $r \in \mathbb{N}$ and an indecomposable projective Λ-module P.

Left Supported Algebras: Components (2)

Proposition 4.

[5, *Corollary* 5.4.] *Let Λ be a representation-infinite left supported algebra. Then the following are equivalent:*

(a) $\mathcal{L}_\Lambda$ *is infinte.*

(b) *There exists a component* Γ *of* $\Gamma(mod\,\Lambda)$ *lying entirely in* $\mathcal{L}_\Lambda$*.*

(c) $\Gamma(mod\,\Lambda)$ *has a postprojective component without injectives.*

Main Result

We are now ready to prove our main theorem.

Theorem 5.

Let Λ be an algebra such that $\mathcal{L}_\Lambda$ or $\mathcal{R}_\Lambda \neq \emptyset$. Then Λ is τ-tilting finite if and only if Λ is representation-finite.

Proof Outline I

- The sufficiency is obvious so we prove the necessity.
- Assume Λ is τ-tilting finite but representation-infinite.
- Without loss of generality, assume $\mathcal{L}_\Lambda \neq \emptyset$.
- Every torsion-free class is functorially finite.
- $(\operatorname{ind}\Lambda \setminus \operatorname{add}\mathcal{L}_\Lambda, \operatorname{add}\mathcal{L}_\Lambda)$ is a torsion pair with $\operatorname{add}\mathcal{L}_\Lambda$ a torsion-free class.
- By Proposition 2 (d), $\operatorname{add}\mathcal{L}_\Lambda = \operatorname{Cogen} I(\operatorname{add}\mathcal{L}_\Lambda)$
- Theorem 3 (b) guarantees Λ is left supported.
- The equivalency of Proposition 4 (a) and (c) guarantees the existence of a postprojective component Γ of $\Gamma(\operatorname{mod}\Lambda)$ with or without injectives.

Proof Outline II

- Since a postprojective component is acyclic, $\operatorname{Hom}_\Lambda(M, \tau M) = 0$ for every indecomposable module $M \in \Gamma$.
- Since Γ is infinite, we have an infinite number of τ-rigid modules which further implies an infinite number of basic τ-tilting modules.
- This is a contradiction to Λ being τ-tilting finite!

Applications I

We recall an algebra Λ is *laura* if the set $\operatorname{ind}\Lambda \setminus (\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$ is finite (see [4] and [12]).

Corollary 6.

Let Λ be a laura algebra. Then Λ is τ-tilting finite if and only if Λ is representation-finite.

Following [3], an algebra Λ is an *ada algebra* if $\Lambda \oplus D\Lambda \in \operatorname{add}(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$. In [2], an algebra Λ is *right ada* if $\Lambda \in \operatorname{add}(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$. Dually, Λ is *left ada* if $D\Lambda \in \operatorname{add}(\mathcal{L}_\Lambda \cup \mathcal{R}_\Lambda)$.

Corollary 7.

If Λ is an ada, right ada, or left ada algebra, then Λ is τ-tilting finite if and only if Λ is representation-finite.

Applications II

The next result gives a necessary and sufficient condition for an indecomposable module Y to be in $\mathcal{L}_\Lambda$.

Theorem 8.

[6, *Theorem* 1.1] *Let Λ be an algebra with $Y \in ind\,\Lambda$. Then $Y \in \mathcal{L}_\Lambda$ if and only if, for every $X \in ind\,\Lambda$ with projective dimension at least two, we have $Hom_\Lambda(X, Y) = 0$.*

The following corollary of Theorem 5 is immediate.

Corollary 9.

Let Λ be an algebra and suppose there exists a simple projective (injective) Λ-module M. Then Λ is τ-tilting finite if and only if Λ is representation-finite.

Applications III

Let Λ be an algebra. We say a Λ-module T is a *tilting module* if $\text{pd}_\Lambda T \leq 1$, $\text{Ext}^1_\Lambda(T,T) = 0$, and the number of non-isomorphic indecomposable summands of T equals the number of non-isomorphic simple Λ-modules. Recall that a tilting module T determines a torsion pair, $(\mathcal{T}(T), \mathcal{F}(T))$, in $\text{mod}\,\Lambda$ where $\mathcal{T}(T)$ (or $\mathcal{F}(T)$) is the full subcategory of those modules M such that $\text{Ext}^1_\Lambda(T,M) = 0$ (or such that $\text{Hom}_\Lambda(T,M) = 0$), (see [7] for details). We say T is *separating* if the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting.

Corollary 10.

Let Λ be an algebra and suppose there exists a non-projective tilting Λ-module T which is separating. Then Λ is τ-tilting finite if and only if Λ is representation-finite.

Applications IIII

The proof outline is as follows:

- Since T is non-projective, $\mathcal{F}(T)$ is non-empty.
- Let $X \in \mathcal{F}(T)$ be indecomposable.
- Since $(\mathcal{T}(T), \mathcal{F}(T))$ is splitting, we know $\tau X \in \mathcal{F}(T)$.
- By the definition of $\mathcal{T}(T)$, each indecomposable injective Λ-module must belong to $\mathcal{T}(T)$.
- $\mathrm{Hom}_\Lambda(I, \tau X) = 0$ for every indecomposable injective I.
- $\mathrm{pd}_\Lambda X \leq 1$.
- Any module Y with $\mathrm{pd}_\Lambda Y \geq 2$ must belong to $\mathcal{T}(T)$.
- $\mathrm{Hom}_\Lambda(Y, X) = 0$ for $X \in \mathcal{F}(T)$.
- Theorem 8 implies $X \in \mathcal{L}_\Lambda$.

Applications IV

We recall that the so-called APR-tilting modules [8] provide a classic sample of separating tilting modules. Let S be a simple projective that is not injective, if it exists, and set $T = \tau^{-1}S \oplus P$ where P is the direct sum of all non-isomorphic indecomposable projective Λ-modules different from S.

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