Grassmanian Categories of Infinite Rank

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Grassmanian Cluster Algebras

Fix $0 < k < n$. Then $\mathrm{Gr}(k,n) =$ space of k-dimensional subspaces of $\mathbb{C}^n$.

It is a projective variety by the Plücker embedding, so we may consider its homogeneous coordinate ring $\mathcal{A}_{k,n} = \mathbb{C}[\mathrm{Gr}(k,n)]$.

Theorem (Scott 2006)

$\mathcal{A}_{k,n}$ has the structure of a cluster algebra.

$$\mathcal{A}_{k,n} \cong \mathbb{C}[p_I \mid I \subset \{1,\dots,n\}, |I| = k]/\mathcal{I}_P$$

where the p_I are called the Plücker coordinates and $\mathcal{I}_P$ is generated by the Plücker relations.

The Plücker coordinates are examples of cluster variables in $\mathcal{A}_{k,n}$.

Compatibility of Plücker coordinates

Definition

Two k-subsets I and J of $\{1, \ldots, n\}$ (or more generally $\mathbb{Z}$) are said to be crossing if there exist $i_1, i_2 \in I \setminus J$ and $j_1, j_2 \in J \setminus I$ such that

$$i_1 < j_1 < i_2 < j_2 \quad \text{or} \quad j_1 < i_1 < j_2 < i_2.$$

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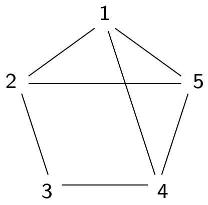
Terminology comes from $k = 2$:

- 2-subsets may be viewed as arcs in an n-gon;
- For example, $n = 5$ and $\{2, 5\}$ and $\{1, 4\}$;
- Here, 'crossing' as defined above corresponds to the arcs actually crossing.



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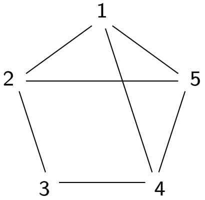

Two Plücker coordinates p_I and p_J of $\mathcal{A}_{k,n}$ are *compatible* if I and J are noncrossing.

Cluster Structure on $\mathcal{A}_{k,n}$

Theorem (Scott 2006)

$\mathcal{A}_{k,n}$ has the structure of a cluster algebra.

$$\mathcal{A}_{k,n} \cong \mathbb{C}[p_I \mid I \subset \{1, \ldots, n\}, |I| = k]/\mathcal{I}_P$$

- Plücker coordinates are examples of cluster variables;
- Maximal sets of compatible Plücker coordinates give examples of clusters;
- If $k = 2$, all cluster variables and clusters arise in this way.

Jensen, King and Su Categorification

Basic Idea: Find an additive category such that:

- indecomposables objects $\leftrightsquigarrow$ cluster variables in $\mathcal{A}_{k,n}$;
- cluster-tilting subcategories $\leftrightsquigarrow$ clusters in $\mathcal{A}_{k,n}$.

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The Grassmanian cluster algebra was first categorified by Geiß, Leclerc and Schröer, but Jensen, King and Su had a different approach using singularities:

- Set $R_{k,n} = \mathbb{C}[x,y]/(x^k - y^{n-k})$ which is an isolated curve singularity;
- The group $\mu_n = \{\zeta \in \mathbb{C} \mid \zeta^n = 1\}$ acts on $R_{k,n}$ via

$$\zeta \cdot x = \zeta x, \quad \zeta \cdot y = \zeta^{-1} y;$$

- Consider $\mathrm{MCM}^{\mu_n} R_{k,n}$ = the category of μ_n-equivariant maximal Cohen-Macaulay $R_{k,n}$-modules.

Jensen, King and Su Categorification

Theorem (Jensen, King and Su 2016)

1. *There is a bijection*

   $$\{\textit{rank one modules in } \mathrm{MCM}^{\mu_n} R_{k,n}\} \leftrightarrow \{\textit{Plücker coordinates in } \mathcal{A}_{k,n}\}.$$

2. *For two rank one modules M and N, $\mathrm{Ext}^1(M, N) = 0$ if and only if the corresponding Plücker coordinates are compatible.*

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Moreover, by showing a relationship between $\mathrm{MCM}^{\mu_n} R_{k,n}$ and the categorification of Geiß, Leclerc and Schröer, they know:

3. *Cluster-tilting subcategories of $\mathrm{MCM}^{\mu_n} R_{k,n}$ exist, and examples of such are given by maximal sets of compatible Plücker coordinates.*
4. *There is a cluster character linking $\mathrm{MCM}^{\mu_n} R_{k,n}$ with the cluster algebra $\mathcal{A}_{k,n}$.*

$k = 2$ or 'Type A'

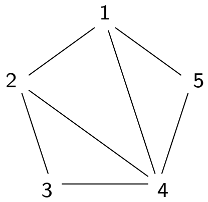
When $k = 2$, $\mathrm{MCM}^{\mu_n} R_{k,n}$ is of finite type i.e. there are finitely many indecomposable objects.

There are bijections between

1. indecomposable objects in $\mathrm{MCM}^{\mu_n} R_{2,n}$;
2. cluster variables in $\mathcal{A}_{2,n}$;
3. arcs in an n-gon.

And further bijections between

1. cluster-tilting subcategories in $\mathrm{MCM}^{\mu_n} R_{2,n}$;
2. clusters in $\mathcal{A}_{2,n}$;
3. triangulations of the n-gon.



Grassmanian Cluster Algebras of Infinite Rank

In 2015, Grabowski and Gratz introduced an infinite version of $\mathcal{A}_{k,n}$:

$$\mathcal{A}_k := \mathbb{C}[p_I \mid I \subset \mathbb{Z}, |I| = k]/\mathcal{I}_P$$

where $\mathcal{I}_P$ is generated by Plücker relations.

- They showed $\mathcal{A}_k$ can be endowed with the structure of a cluster algebra in infinitely many ways;
- Gratz also showed that $\mathcal{A}_k$ is the colimit of the cluster algebras $\mathcal{A}_{k,n}$ in the category of rooted cluster algebras;
- Groechenig further showed that $\mathcal{A}_k$ is isomorphic to the coordinate ring of an infinite rank Grassmanian.

Grassmanian Categories of Infinite Rank

Idea: Take $n \to \infty$ in the work of Jensen, King and Su:

- The singularity:

$$R_{k,n} = \mathbb{C}[x,y]/(x^k - y^{n-k}) \quad \rightsquigarrow \quad R_k = \mathbb{C}[x,y]/(x^k);$$

- The group action:

$$\mu_n \curvearrowright R_{k,n} \quad \rightsquigarrow \quad \begin{aligned} \mathbb{G}_m = \mathbb{C}^* &\curvearrowright R_k, \\ \zeta \cdot x &= \zeta x, \\ \zeta \cdot y &= \zeta^{-1} y; \end{aligned}$$

- The category: $\mathrm{MCM}^{\mu_n} R_{k,n} \quad \rightsquigarrow \quad \mathrm{MCM}^{\mathbb{G}_m} R_k.$

Grassmanian Categories of Infinite Rank

But as the character group of $\mathbb{G}_m$ is $\mathbb{Z}$, there is an equivalence of categories

$$\mathrm{MCM}^{\mathbb{G}_m} R_k \simeq \mathrm{MCM}_{\mathbb{Z}} R_k$$

where the latter is the category of $\mathbb{Z}$-graded MCM R_k modules, with $|x| = 1$ and $|y| = -1$.

Definition

We call $\mathrm{MCM}_{\mathbb{Z}} R_k$ the Grassmanian category of type (k, ∞).

What do we know about this category?

- R_k is a non-isolated hypersurface singularity, and hence is Gorenstein and $\mathrm{MCM}_{\mathbb{Z}} R_k$ is a Frobenius category.
- When $k = 2$, this is the curve singularity of type A_∞:
  - By Buchweitz–Greuel–Schreyer, we know all indecomposable objects:

$$\begin{array}{rll} (x, y^i)(j) & \text{where} & i \geq 0, j \in \mathbb{Z} \\ \mathbb{C}[y](\ell) & \text{where} & \ell \in \mathbb{Z} \end{array}$$

  - Our category is related to others in the literature studying cluster combinatorics of type A_∞: Holm–Jørgensen, Paquette–Yildirum.
- However, when $k \geq 3$, $\mathrm{MCM}_{\mathbb{Z}} R_k$ is wild.

Generalising rank one modules

Recall that JKS gave a bijection

$$\{\text{rank one modules in } \mathrm{MCM}^{\mu_n} R_{k,n}\} \leftrightarrow \{\text{Plücker coordinates in } \mathcal{A}_{k,n}\}.$$

We would like to replicate this, but as R_k is not reduced, we need to be more careful what we mean by "rank".

Definition

Let $\mathcal{F} = \mathbb{C}[x, y^{\pm}]/(x^k)$ be the total ring of fractions for R_k. Then we say $M \in \mathrm{MCM}_{\mathbb{Z}} R_k$ is generically free of rank n if $M \otimes_{R_k} \mathcal{F}$ is a free $\mathcal{F}$-module of rank n.

Classifying generically free modules

Proposition (ACFGS)

1. *If $M \in \mathrm{MCM}_{\mathbb{Z}} R_k$ is generically free then $M \cong \Omega(N)$ for some finite dimensional (over $\mathbb{C}$) graded R_k-module N.*
2. *$M \in \mathrm{MCM}_{\mathbb{Z}} R_k$ is generically free of rank one $\iff$ M is isomorphic to a shift of a graded ideal of R_k which contains a power of y.*
3. *Every homogeneous ideal of R_k can be generated by monomials.*

Corollary (ACFGS)

A module $M \in \mathrm{MCM}_{\mathbb{Z}} R_k$ is generically free of rank one $\iff$ M is isomorphic to

$$(x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \ldots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$$

for some $0 \leq i_1 \leq i_2 \leq \cdots \leq i_{k-2} \leq i_{k-1}$ and $i_k \in \mathbb{Z}$.

Connection to Plücker coordinates

Consider $k = 4$ and $I = (x^3, x^2y^2, xy^2, y^4)(1)$ - how do we get a 4-subset?

$\deg_I$:	-5	-4	-3	-2	-1	0	1	2
$\cdots$	x^3y^7	x^3y^6	x^3y^5	x^3y^4	x^3y^3	x^3y^2	x^3y	x^3
$\cdots$	x^2y^6	x^2y^5	x^2y^4	x^2y^3	x^2y^2	x^2y	x^2	
$\cdots$	xy^5	xy^4	xy^3	xy^2	xy	x		
$\cdots$	y^4	y^3	y^2	y	1			

Look at where the rows end - $\ell(I) = (-5, -2, -1, 2)$

This equivalent to $\ell(I) = \left(\deg_I(y^4), \deg_I(xy^2), \deg_I(x^2y^2), \deg_I(x^3)\right)$.

Set $\ell(I) = \left(\deg_I(y^{i_{k-1}}), \deg_I(xy^{i_{k-2}}), \ldots, \deg_I(x^{k-2}y^{i-1}), \deg_I(x^{k-1})\right)$.

- This gives a strictly increasing k-subset;
- $\deg_I(x^{k-1}) = k - 1 - i_k$, so we can immediately recover i_k (the shift of the ideal I) from the last term of $\ell(I)$;
- we may also recover each i_j from $\ell(I)_{k-j} = k - j - 1 - i_j - i_k$.

Theorem (ACFGS)

There is a bijection

$$\left\{\begin{matrix} \textit{generically free modules of} \\ \textit{rank one in } \mathrm{MCM}_{\mathbb{Z}} R_k \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{Plücker coordinates} \\ \textit{in } \mathcal{A}_k \end{matrix}\right\}$$

$$I \mapsto p_{\ell(I)}.$$

Moreover, $\mathrm{Ext}^1(I, J) = 0$ *if and only if* $p_{\ell(I)}$ *and* $p_{\ell(J)}$ *are compatible (or equivalently* $\ell(I)$ *and* $\ell(J)$ *are noncrossing).*

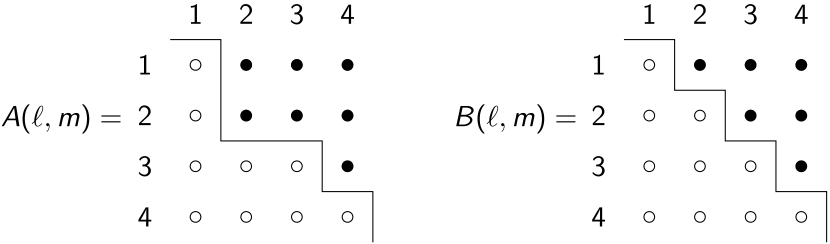
Our combinatorial tool

Associated to two k-subsets $\ell = (\ell_1, \dots, \ell_k)$ and $m = (m_1, \dots, m_k)$, we get two staircase paths in a $(k \times k)$ grid:

- Both paths go from the top left to the bottom right;
- For path A (respectively path B), a box (i, j) lies above the path if and only if $\ell_i \leq m_j$ (respectively $\ell_i < m_j$).

Take $k = 4$ and consider the subsets ℓ and m with

$$m_1 < \ell_1 < \ell_2 = m_2 < m_3 < \ell_3 < m_4 < \ell_4.$$



If ℓ and m are disjoint then:

- $A(\ell, m) = B(\ell, m)$;
- we can describe the path by reading from smallest to largest:
    - each time you read an m go right;
    - each time you read an ℓ go down.

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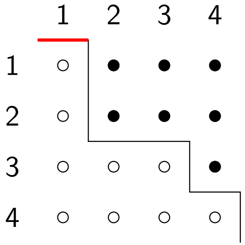

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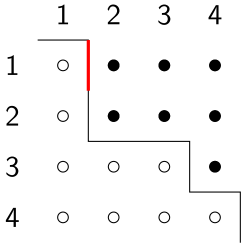

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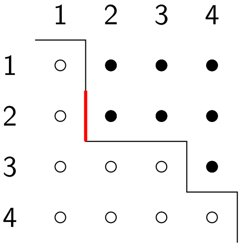

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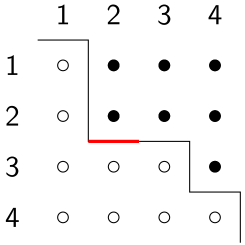

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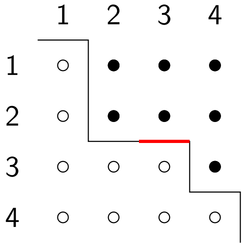

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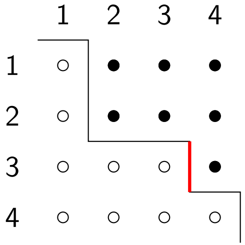

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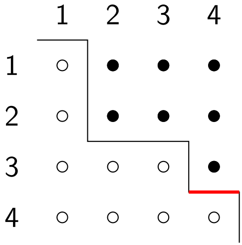

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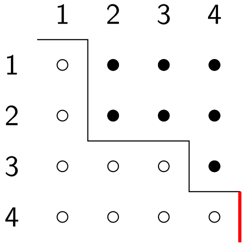

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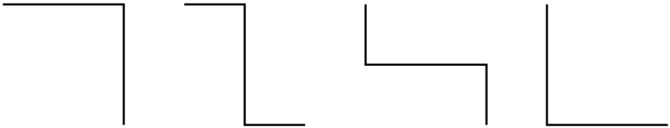
- we can describe the path by reading from smallest to largest:
  - each time you read an m go right;
  - each time you read an ℓ go down.
- We can also read the number of 'crossings' between ℓ and m using the number of steps.
- In particular, ℓ and m are noncrossing if and only if the staircase path has a single step:

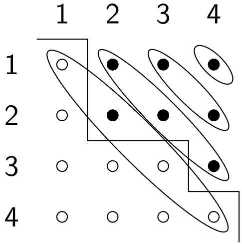


From these staircases, we extract two numbers:

Definition

Let $\alpha(\ell, m)$ be the number of upper diagonals that lie completely above the staircase path in $A(\ell, m)$.

$A(\ell, m) =$



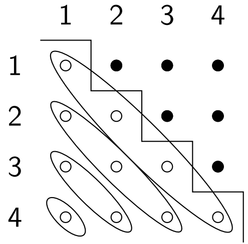
$\Longrightarrow \alpha(\ell, m) = 3$

Similarly:

Definition

Let $\beta(\ell, m)$ be the number of lower diagonals that lie completely below the staircase path in $B(\ell, m)$.

$B(\ell, m) =$



$\Longrightarrow \beta(\ell, m) = 4$

Theorem (ACFGS)

If ℓ and m are two k-subsets, then ℓ and m are noncrossing if and only if

$$\alpha(\ell, m) + \beta(\ell, m) - |\ell \cap m| = k.$$

Easy to show when ℓ and m are disjoint (using the single step pictures) - then use induction by removing the common terms, and showing how α and β change.

Example: For $m_1 < \ell_1 < \ell_2 = m_2 < m_3 < \ell_3 < m_4 < \ell_4$, we have

$$\alpha(\ell, m) + \beta(\ell, m) - |\ell \cap m| = 3 + 4 - 1 = 6 \neq 4,$$

and we see that there is a crossing $m_1 < \ell_1 < m_3 < \ell_3$.

Connection to Ext dimension

Take two generically free modules of rank one in $\mathrm{MCM}_{\mathbb{Z}} R_k$, say I and J. Then to calculate $\mathrm{Ext}^1(I, J)$, use the matrix factorisation of I

$$R_k^k \xrightarrow{M} R_k^k \xrightarrow{N} R_k^k \to I \to 0$$

to give a graded projective presentation of I. Apply $\mathrm{grHom}(-, J)$, noting that $\mathrm{grHom}(R_k(m), J) \cong J(-m)$ to get

$$\mathbb{J} \xrightarrow{N^T} \mathbb{J}(1) \xrightarrow{M^T} \mathbb{J}(k)$$

where each $\mathbb{J}$ is a direct sum of k appropriately shifted copies of J. Then

$$\mathrm{Ext}^1(I, J) = (\ker(M^T))_0 / (\mathrm{im}(N^T))_0.$$

Then, simply using rank-nullity we may show

$$\begin{aligned}\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J)) &= \dim_{\mathbb{C}}((\ker(M^T))_0) - \dim_{\mathbb{C}}((\mathrm{im}(N^T))_0) \\ &= \Big(\dim_{\mathbb{C}}(\mathbb{J}(1)_0) - \dim_{\mathbb{C}}(\mathrm{im}(M^T))_0)\Big) \\ &\quad - \Big(\dim_{\mathbb{C}}(\mathbb{J}_0) - \dim_{\mathbb{C}}(\ker(N^T))_0)\Big)\end{aligned}$$

Then, simple calculations using the matrices M and N shows

$$\begin{aligned}\dim_{\mathbb{C}}(\mathbb{J}_0) - \dim_{\mathbb{C}}(\mathbb{J}(1)_0) &= |\ell(I) \cap \ell(J)| \\ \dim_{\mathbb{C}}(\mathrm{im}(M^T))_0) &= k - \beta(\ell(I), \ell(J)) \\ \dim_{\mathbb{C}}(\ker(N^T))_0) &= \alpha(\ell(I), \ell(J))\end{aligned}$$

Theorem (ACFGS)

$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I,J)) = \alpha(\ell(I), \ell(J)) + \beta(\ell(I), \ell(J)) - k - |\ell(I) \cap \ell(J)|.$

Combining the results

Theorem (ACFGS)

If ℓ and m are two k-subsets, then ℓ and m are noncrossing if and only if

$$\alpha(\ell, m) + \beta(\ell, m) - |\ell \cap m| = k.$$

Theorem (ACFGS)

$$\dim_{\mathbb{C}}(\mathrm{Ext}^1(I, J)) = \alpha(\ell(I), \ell(J)) + \beta(\ell(I), \ell(J)) - k - |\ell(I) \cap \ell(J)|.$$

Corollary (ACFGS)

If I and J are two generically free modules of rank 1 in $\mathrm{MCM}_{\mathbb{Z}} R_k$ then $\mathrm{Ext}^1(I, J) = 0$ if and only if $\ell(I)$ and $\ell(J)$ are noncrossing.

$k = 2$ case

Recall that when $k = 2$, all indecomposable objects are of the form:

$$\begin{aligned} (x, y^i)(j) \quad &\text{where} \quad i \geq 0, j \in \mathbb{Z} \\ \mathbb{C}[y](\ell) \quad &\text{where} \quad \ell \in \mathbb{Z}. \end{aligned}$$

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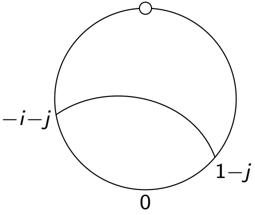
$$\mathbb{C}[y](\ell) \quad \text{where} \quad \ell \in \mathbb{Z}.$$

The $(x, y^i)(j)$ are the generically free modules, which are all of rank 1.

They can be classified by arcs in an ∞-gon:

$$(x, y^i)(j) \quad \to \quad (-i-j, 1-j) \quad \to$$



$k = 2$ case

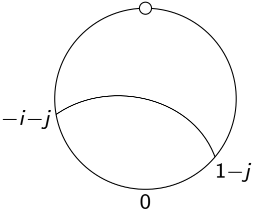
Recall that when $k = 2$, all indecomposable objects are of the form:

$$\begin{array}{rcl} (x, y^i)(j) & \text{where} & i \geq 0, j \in \mathbb{Z} \\ \mathbb{C}[y](\ell) & \text{where} & \ell \in \mathbb{Z}. \end{array}$$

The $(x, y^i)(j)$ are the generically free modules, which are all of rank 1.

They can be classified by arcs in an ∞-gon:

$$(x, y^i)(j) \quad \to \quad (-i - j, 1 - j) \quad \to$$



Sets of noncrossing arcs correspond to rigid subcategories of $\mathrm{MCM}_{\mathbb{Z}} R_2$.

Holm and Jørgensen cluster combinatorics of A_∞

These are the same combinatorics studied by Holm and Jørgensen. They consider the category

$$D^f_{dg}(\mathbb{C}[y])$$

i.e. the derived category of dg modules with finite dimensional homology over the dga $\mathbb{C}[y]$ with zero differential.

$$\text{indecomposable objects} \longleftrightarrow \text{arcs in an } \infty\text{-gon}$$

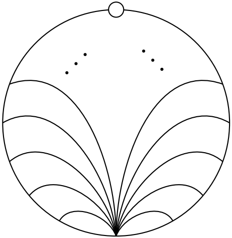
$$\text{maximal rigid subcategories} \longleftrightarrow \text{triangulations in an } \infty\text{-gon}$$

Let $\mathcal{C}$ be the full subcategory of $\mathrm{MCM}_{\mathbb{Z}} R_2$ generated by generically free modules. Then

$$\underline{\mathcal{C}} \simeq D^f_{dg}(\mathbb{C}[y]).$$

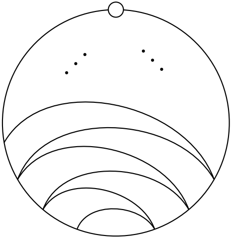
Holm and Jørgensen cluster combinatorics of A_∞

Since $\underline{\mathcal{C}} \simeq D^f_{dg}(\mathbb{C}[y])$, the cluster-tilting subcategories in both are the same and by Holm–Jørgensen, these correspond to triangulations of the ∞-gon containing either:



a fountain

or

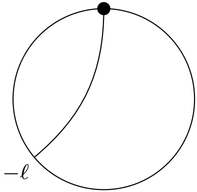


a leap frog

What about the other modules?

We can also include the modules $\mathbb{C}[y](\ell)$ in the combinatorial model by completing the ∞-gon i.e. adding a point at $-\infty$.

$\mathbb{C}[y](\ell) \leftrightarrow (-\infty, -\ell)$

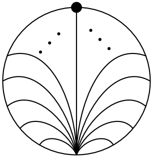


- If (a, b) is a finite arc and $(-\infty, -\ell)$ is an infinite arc, then Ext^1 vanishes between the corresponding modules if and only if the arcs are noncrossing.
- $$\mathrm{Ext}^1(\mathbb{C}[y], \mathbb{C}[y](\ell)) = \begin{cases} \mathbb{C} & \text{if } \ell < 0, \\ 0 & \text{otherwise.} \end{cases}$$
- So maximal rigid subcategories in $\mathrm{MCM}_{\mathbb{Z}} R_2$ are maximal sets of noncrossing arcs with at most one infinite arc.

Cluster-tilting subcategories

Theorem (ACFGS)

The cluster-tilting subcategories of $\mathrm{MCM}_{\mathbb{Z}} R_2$ *correspond precisely to maximal sets of noncrossing arcs in the completed* ∞*-gon, which contain a fountain.*



Using this combinatorial model, we are able to see connections to other work in the literature.

Proposition (ACFGS)

$\mathrm{MCM}_{\mathbb{Z}} R_2$ *is equivalent to the completed discrete cluster category of infinite type corresponding to a disk with a single accumulation point, as studied by Paquette–Yildirum.*

Thank you!