Pointed Hopf algebras of discrete (co)representation type

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(Joint with M. Iovanov, E.Sen, A. Sistko)

FD seminar

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Notations:

- \( \mathbb{K} \) is an algebraically closed field with \( \text{char} \mathbb{K} = 0 \).
- An algebra \( A \) is basic if simple \( A \)-modules are 1 dimensional over \( \mathbb{K} \).
- A coalgebra \( C \) is pointed if simple \( C \)-comodules are 1-dimensional over \( \mathbb{K} \).
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\[ \{ \text{f.d. pointed coalgebras} \} \xrightarrow{\text{Hom}(\cdot,\mathbb{K})} \{ \text{f.d. basic algebras} \} \]
Path coalgebra:

Let $Q = (Q_0, Q_1)$ be a quiver. The path coalgebra $\mathbb{K}Q$ is spanned by all the paths in $Q$ with comultiplication $\Delta(p) = \sum_{p = \langle p_1 | p_2 \rangle} p_1 \otimes p_2$; counit $\epsilon(e_i) = 1$ and $\epsilon(p) = 0$ for $|p| > 0$. 

Rmks: 1. We use the notation $\mathbb{K}[Q]$ for path algebras.
2. For finite acyclic quiver $Q$, $\mathbb{K}[Q]^\ast = \mathbb{K}Q^\text{op}$.
3. But usually the algebra structure of $\mathbb{K}[Q]$ and coalgebra structure of $\mathbb{K}Q$ is not compatible. Hence cannot form a "path bialgebra."
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Example: $Q : 3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$.

$\Delta(\langle \alpha | \beta \rangle) = e_3 \otimes \langle \alpha | \beta \rangle + \alpha \otimes \beta + \langle \alpha | \beta \rangle \otimes e_1$
Path coalgebra:

Let $Q = (Q_0, Q_1)$ be a quiver. The path coalgebra $KQ$ is spanned by all the paths in $Q$ with comultiplication $\Delta(p) = \sum_{p=\langle p_1 | p_2 \rangle} p_1 \otimes p_2$; counit $\epsilon(e_i) = 1$ and $\epsilon(p) = 0$ for $|p| > 0$.

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Rmks: 1. We use the notation $K[Q]$ for path algebras.
2. For finite acyclic quiver $Q$, $K[Q]^* = KQ^{op}$.
3. But usually the algebra structure of $K[Q]$ and coalgebra structure of $KQ$ is not compatible. Hence cannot form a “path bialgebra”.

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Theorem (Gabriel)

A basic algebra $A$ is isomorphic to a quiver algebra $\mathbb{K}[Q]/I$ for some admissible ideal $I$.

Dually,

Theorem (Woodcock, 97)

A pointed coalgebra $C$ is isomorphic to an admissible subcoalgebra of a path coalgebra $\mathbb{K}Q$. 
Let $C$ be a coalgebra
Group-like elements $G(C) = \{g \in C | \Delta(g) = g \otimes g\}$.
Skew primitive elements $P(g, h) = \{x | \Delta(x) = g \otimes x + x \otimes h\}$, where $g, h \in G(C)$.
$x \in P(g, h)$ is trivial if $x = k(g - h)$ for some $k \in \mathbb{K}$. 

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**Definition**

*For a pointed coalgebra $C$, define its Ext-quiver $Q$ as the following: Vertices = group-likes $g$; Number of arrows $g \to h = \dim_{\mathbb{K}} P(g, h) - 1$.*
Example: Taft algebra $T_n = \langle g, x | g^n = 1, x^n = 0, gxg^{-1} = qx \rangle$, where $q$ is a primitive $n - \text{th}$ root of unity. The coalgebra structure is given by $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$.

The Ext quiver $Q$ of $T_n$ is
Representation types

- An algebra $A$ is finite representation type if there are only finitely many isomorphism classes of indecomposable $A$-modules.
- A coalgebra $C$ is finite corepresentation type if there are only finitely many isomorphism classes of indecomposable $C$-comodules.
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(For a finite dimensional coalgebra $C$, $C$ is finite corepresentation type if and only if $C^*$ is a finite representation type algebra)

• A Hopf algebra $H$ is finite (co)-representation type if as a (co)-algebra $H$ is finite (co)-representation type.
Some known results about finite type Hopf algebras:
When $H = kG$ for some finite group $G$ over an algebraically closed field $k$. 
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- [Maschke, 1899] When $\text{char } k \nmid |G|$, $kG$ is semisimple. Hence it is finite representation type.

  $\#$ indecomposable $kG$-modules $= \#$ conjugacy classes of $G$. 
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  \# indecomposable $kG$-modules $= \#$ conjugacy classes of $G$.

- [D.G.Higman 1954] When $p = \text{char } k \mid |G|$, $kG$ is representation finite type if and only if Sylow $p$ subgroups are cyclic.
Classification of finite-dimensional monomial Hopf algebras (over $k$ containing all roots of unity, $\text{char} k = 0$):


A Hopf algebra is monomial if and only if it is basic and Nakayama.
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Classification of finite-dimensional monomial Hopf algebras
(over $k$ with $\text{char } k = p$):
Classification of finite-dimensional (pointed) basic Hopf algebras of finite (co)representation type (over an algebraically closed field $k$).


A basic Hopf algebra is finite representation type if and only if it is Nakayama.
Next we consider infinite-dimensional pointed Hopf algebras.

**Definition**

Let $C$ be a pointed coalgebra. We say that $C$ is of discrete corepresentation type, if for any finite dimension vector $d$, there are only finitely many isoclasses of $C$-comodules of dimension vector $d$. 
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**Definition**

Let $C$ be a pointed coalgebra. We say that $C$ is of discrete corepresentation type, if for any finite dimension vector $d$, there are only finitely many isoclasses of $C$-comodules of dimension vector $d$.

Rmks: 1. For finite-dimensional coalgebras, Brauer-Thrall conjecture $\implies$ discrete type$=$finite type.
2. $C$ is of discrete corepresentation type if and only if any finite dimensional subcoalgebra $D \subseteq C$ is finite corepresentation type.
Classification of coserial \(^1\) pointed Hopf algebras (over a field \(k\) containing all roots of unity).


\(^1\)A Hopf algebra \(H\) is coserial = \(H\) is a serial coalgebra = every f.d. indecomposable \(H\)-comodule is uniserial.
Classification of coserial $^1$ pointed Hopf algebras (over a field $k$ containing all roots of unity).


The Ext quiver of a coserial pointed Hopf algebra is one of the following:

1. copies of a single vertex
2. copies of a complete oriented cycle,
3. copies of an infinite quiver

$\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots$.

---

$^1$A Hopf algebra $H$ is coserial $\iff$ $H$ is a serial coalgebra $\iff$ every f.d. indecomposable $H$-comodule is uniserial.
Let $H$ be a discrete corepresentation type pointed Hopf algebra over $\mathbb{K}$. First we want to classify all the possible Ext quivers $Q$ of $H$. 
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**Lemma**

Let $(H, m, u, \Delta, \epsilon, S)$ be a pointed Hopf algebra and $x \in P(1, a)$ be a skew primitive. Then

1. (Translation) For any group like $g \in G(H)$, $gx \in P(g, ga)$.
2. $S(x) = -xa^{-1} \in P(a^{-1}, 1)$. 

Proof.  

(1) $\Delta(gx) = \Delta(g)\Delta(x) = (g \otimes g)(1 \otimes x + x \otimes a) = gx \otimes gx + gx \otimes ga$.  

Apply the axiom for antipode $m(1 \otimes S)\Delta = \epsilon$ to $x$. 

Let \( H \) be a discrete corepresentation type pointed Hopf algebra over \( \mathbb{K} \). First we want to classify all the possible Ext quivers \( Q \) of \( H \).

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**Proof.**

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Corollary

If $H$ is a pointed Hopf algebra, then its Ext quiver $Q$ is homogeneous. i.e. for each vertex $v$

# arrows coming out of $v$ = # arrows going into $v$ = $N$. 

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Corollary

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# arrows coming out of $v$ = # arrows going into $v$ = $N$.

If $H$ is discrete corepresentation type, then $Q$ must be Schurian. i.e. no multiple arrows between any two vertices. Otherwise,

\[
\bullet \quad \Rightarrow \quad \bullet \quad \subseteq \quad H
\]

\[\implies H \text{ is not discrete corepresentation type}.\]
If $N \geq 4$ and no loop, then $\cup H \notin H \Rightarrow H$ is not discrete corepresentation type.
If there are two arrows $a \xleftarrow{\times} 1 \xrightarrow{\cdot} b$, then $ab = ba$. Otherwise, by translation

$$
\begin{array}{c}
a \\
\downarrow ay \\
ab
\end{array}
\begin{array}{c}
\downarrow xa \\
ba
\end{array}
\begin{array}{c}
\downarrow xb \\
ya
\end{array}
\begin{array}{c}
b \\
\downarrow \subset H
\end{array}

\implies H \text{ is not discrete corepresentation type.}
Proposition

If $H$ is discrete corepresentation type, then $N < 3$. 

Proof. If there are 3 outgoing arrows from 1 say to $a$, $b$, $c$, then $a\xrightarrow{ab} b\xrightarrow{bc} c\xrightarrow{ca} a \subseteq H$.

Case 1. If all vertices are mutually distinct: $= \Rightarrow H$ is not discrete corepresentation type.

Case 2. Not vertices are mutually distinct: Use "covering map" of coalgebras and reduce to Case 1.
Proposition

If $H$ is discrete corepresentation type, then $N < 3$.

Proof. If there are 3 outgoing arrows from 1 say to $a$, $b$, $c$, then

$$
\begin{array}{c}
a \rightarrow ay \quad ab \\
z \quad \quad \quad xb \\
ca \\
x \quad \quad \quad bz \\
c \rightarrow yc \quad bc
\end{array}
$$

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Proposition

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![Diagram showing arrows between vertices labeled $a, b, c$ and $ab, bc, ca$]

Case 1. If all vertices are mutually distinct: $\implies H$ is not discrete corepresentation type.
Proposition

*If* \( H \) *is discrete corepresentation type, then* \( N < 3 \).

**Proof.** If there are 3 outgoing arrows from 1 say to \( a, b, c \), then

\[
\begin{align*}
  a & \rightarrow ab \\
  z\uparrow & \quad \downarrow x \quad b \\
  ca & \quad \downarrow \quad b \quad \downarrow \quad bc \\
  c & \quad \rightarrow yc
\end{align*}
\]

Case 1. If all vertices are mutually distinct: \( \implies \) \( H \) is not discrete corepresentation type.

Case 2. Not vertices are mutually distinct: Use “covering map” of coalgebras and reduce to Case 1.
Definitions:

- A (s-t)-diamond in $C \subseteq \mathbb{K}Q$ is a linear combination of paths starting from $s$ and ending in $t$.
- A diamond basis of $C$ is a basis containing only diamonds as well as containing all vertices and arrows.
- Any finite dimensional pointed coalgebra has a diamond basis [JMR].
- A covering map $f : C \rightarrow D$ is a coalgebra homomorphism, which (1) sends a diamond basis of $C$ to a diamond basis of $D$; (2) sends diamonds sharing same start vertex or terminal vertex to the same diamond.
- If $f : C \rightarrow D$ is a covering map, then $f^* : D^* \rightarrow C^*$ is a separable extension of algebras. Hence preserving finite representation types [IS, ISSZ].
Theorem (Iovanov, Sen, Sistko, Zhu)

If $H$ is a connected pointed Hopf algebras of discrete representation type, then the Ext quiver of $H$ is one of following:

(0) A single vertex.

(1) A complete oriented cycle;

(2) \[ \cdots \rightarrow \cdot \rightarrow \cdot \rightarrow \cdots; \]

(3) \[ \cdots \begin{array}{ccc}
\vdots & \vdots & \vdots \\
\vdots & y^2 \rightarrow b^2 & \rightarrow \cdot \\
\vdots & y \rightarrow b & \rightarrow ab \\
\vdots & x \rightarrow 1 & \rightarrow a \\
\vdots & x^2 \rightarrow a^2 & \rightarrow \cdot \\
\vdots & \vdots & \vdots \\
\end{array} \rightarrow \cdots \]
(4) The quiver in (3) identifying vertices $a^m = b^n$. (The quiver looks like a tube.)
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Computing algebra structures for case (3) and (4):

$H_{m,n}^m(\lambda, s, t, k)$ is generated by $a, b, x, y$ satisfying the following conditions, where $\lambda \neq 0, s, t, k \in \mathbb{K}$ ($\mathbb{K}$ algebraically closed, char $\mathbb{K}=0$).

\[
\begin{align*}
ab &= ba, \quad a^m = b^n, \quad xy + \lambda yx = k(1 - ab), \\
ax + xa &= 0, \quad \lambda bx + xb = 0, \quad x^2 = s(1 - a^2), \\
by + yb &= 0, \quad ay + \lambda ya = 0, \quad y^2 = t(1 - b^2); \\
\Delta(a) &= a \otimes a, \quad \Delta(b) = b \otimes b, \\
\Delta(x) &= 1 \otimes x + x \otimes a, \quad \Delta(y) = 1 \otimes y + y \otimes b; \\
\epsilon(a) &= \epsilon(b) = 1, \quad \epsilon(x) = (y) = 0; \\
S(a) &= a^{-1}, \quad S(b) = b^{-1}, \quad S(x) = -xa^{-1}, \quad S(y) = -yb^{-1}.
\end{align*}
\]


