Pointed Hopf algebras of discrete (co)representation type

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Notations:

- $\mathbb{K}$ is an algebraically closed field with $char\mathbb{K} = 0$.
- An algebra A is basic if simple A-modules are 1 dimensional over $\mathbb{K}$.
- A coalgebra C is pointed if simple C-comodules are 1-dimensional over $\mathbb{K}$.

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$$\{\text{ f.d. pointed coalgebras}\} \overset{\mathrm{Hom}(-,\mathbb{K})}{\longleftrightarrow} \{\text{f.d. basic algebras }\}$$

Path coalgebra:

Let $Q = (Q_0, Q_1)$ be a quiver. The path coalgebra $\mathbb{K}Q$ is spanned by all the paths in Q with comultiplication $\Delta(p) = \sum_{p=\langle p_1|p_2\rangle} p_1 \otimes p_2$; counit $\epsilon(e_i) = 1$ and $\epsilon(p) = 0$ for $|p| > 0$.

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Example: $Q : 3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$.
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Rmks: 1. We use the notation $\mathbb{K}[Q]$ for path algebras.

2. For finite acyclic quiver Q, $\mathbb{K}[Q]^* = \mathbb{K}Q^{op}$.

3. But usually the algebra structure of $\mathbb{K}[Q]$ and coalgebra structure of $\mathbb{K}Q$ is not compatible. Hence cannot form a "path bialgebra".

Theorem (Gabriel)

A basic algebra A is isomorphic to a quiver algebra $\mathbb{K}[Q]/I$ for some admissible ideal I.

Dually,

Theorem (Woodcock, 97)

A pointed coalgebra C is isomorphic to an admissible subcoalgebra of a path coalgebra $\mathbb{K}Q$.

Let C be a coalgebra

Group-like elements $G(C) = \{g \in C | \Delta(g) = g \otimes g\}$.

Skew primitive elements $P(g,h) = \{x | \Delta(x) = g \otimes x + x \otimes h\}$, where $g, h \in G(C)$.

$x \in P(g,h)$ is trivial if $x = k(g-h)$ for some $k \in \mathbb{K}$.

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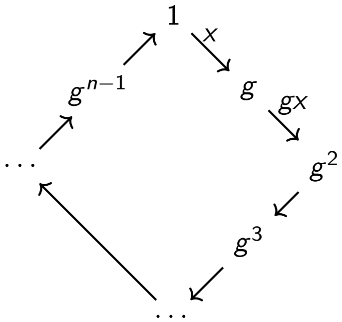
Definition

For a pointed coalgebra C, define its Ext-*quiver Q as the following:*
Vertices= group-likes g;
Number of arrows $g \to h = \dim_{\mathbb{K}} P(g, h) - 1$.

Example: Taft algebra $T_n =< g, x | g^n = 1, x^n = 0, gxg^{-1} = qx >$, where q is a primitive $n-th$ root of unity. The coalgebra structure is given by $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$.
The Ext quiver Q of T_n is



Representation types

- An algebra A is finite representation type if there are only finitely many isomorphism classes of indecomposable A-modules.
- A coalgebra C is finite corepresentation type if there are only finitely many isomorphism classes of indecomposable C-comodules.

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(For a finite dimensional coalgebra C, C is finite corepresentation type if and only if C^* is a finite representation type algebra)

• A Hopf algebra H is finite (co)-representation type if as a (co)-algebra H is finite (co)-representation type.

Some known results about finite type Hopf algebras:
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  # indecomposable kG-modules= # conjugacy classes of G.
- [D.G.Higman 1954] When $p =$char $k \mid |G|$, kG is representation finite type if and only if Sylow p subgroups are cyclic.

- Classification of finite-dimensional monomial Hopf algebras (over k containing all roots of unity, $\mathrm{char} k = 0$):
  X-W. Chen, H-L. Huang, Y. Ye, and P. Zhang, *Monomial Hopf algebras*, 2004.
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- Classification of finite-dimensional monomial Hopf algebras (over k with $\mathrm{char} k = p$):
  G-X. Liu, and Y. Ye, *Monomial Hopf algebras over fields of positive characteristic*, 2006.

- Classification of finite-dimensional (pointed) basic Hopf algebras of finite (co)representation type (over an algebraically closed field k).
  G-X. Liu, F. Li, *Pointed Hopf algebras of finite corepresentation type and their classifications*, 2007.
  A basic Hopf algebra is finite representation type if and only if it is Nakayama.

Next we consider infinite-dimensional pointed Hopf algebras.

Definition

Let C be a pointed coalgebra. We say that C is of discrete corepresentation type, if for any finite dimension vector $\underline{d}$, there are only finitely many isoclasses of C-comodules of dimension vector $\underline{d}$.

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Rmks: 1. For finite-dimensional coalgebras, Brauer-Thrall conjecture $\implies$ discrete type=finite type.
2. C is of discrete corepresentation type if and only if any finite dimensional subcoalgebra $D \subseteq C$ is finite corepresentation type.

- Classification of coserial [1] pointed Hopf algebras (over a field k containing all roots of unity).
  M. C. Iovanov, *Infinite dimensional serial algebras and their representations*, 2018.

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- Classification of coserial [1] pointed Hopf algebras (over a field k containing all roots of unity).
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  The Ext quiver of a coserial pointed Hopf algebra is one of the following:
  (1) copies of a single vertex
  (2) copies of a complete oriented cycle,
  (3) copies of an infinite quiver
  $$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots.$$

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Let H be a discrete corepresentation type pointed Hopf algebra over $\mathbb{K}$. First we want to classify all the possible Ext quivers Q of H.

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Lemma

Let $(H, m, u, \Delta, \epsilon, S)$ be a pointed Hopf algebra and $x \in P(1, a)$ be a skew primitive. Then
(1) (Translation) For any group like $g \in G(H)$, $gx \in P(g, ga)$.
(2) $S(x) = -xa^{-1} \in P(a^{-1}, 1)$.

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Proof.

(1)$\Delta(gx) = \Delta(g)\Delta(x) = (g \otimes g)(1 \otimes x + x \otimes a) = g \otimes gx + gx \otimes ga$.
(2) Apply the axiom for antipode $m(1 \otimes S)\Delta = \epsilon$ to x.

□

Corollary

If H is a pointed Hopf algebra, then its Ext quiver Q is homogeneous. i.e. for each vertex v
$\#$ arrows coming out of $v = \#$ arrows going into $v = N$.

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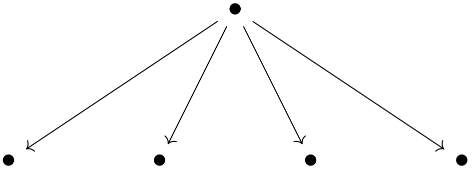
$\#$ arrows coming out of $v = \#$ arrows going into $v = N$.

If H is discrete corepresentation type, then Q must be Schurian. i.e. no multiple arrows between any two vertices. Otherwise,

$$\bullet \rightrightarrows \bullet \quad \subseteq \quad H$$

$\implies$ H is not discrete corepresentation type.

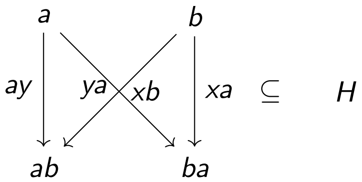
If $N \geq 4$ and no loop, then



$\subseteq \quad H$

$\Longrightarrow$ H is not discrete corepresentation type.

If there are two arrows $a \xleftarrow{x} 1 \xrightarrow{y} b$, then $ab = ba$.
Otherwise, by translation



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Proposition

If H is discrete corepresentation type, then $N < 3$.

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Proof. If there are 3 outgoing arrows from 1 say to a, b, c, then

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\begin{array}{ccccc}
 & a & \xrightarrow{ay} & ab & \\
{}^{za}\swarrow & & & & \nwarrow^{xb} \\
ca & & & & b \qquad \subseteq \qquad H \\
{}^{cx}\nwarrow & & & & \swarrow^{bz} \\
 & c & \xrightarrow[yc]{} & bc &
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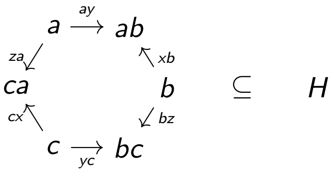

Case 1. If all vertices are mutually distinct: $\Longrightarrow$ H is not discrete corepresentation type.

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Proof. If there are 3 outgoing arrows from 1 say to a, b, c, then

$$\begin{array}{ccccc} & a & \xrightarrow{ay} & ab & \\ {}^{za}\swarrow & & & & \nwarrow^{xb} \\ ca & & & & b \\ {}^{cx}\nwarrow & & & & \swarrow^{bz} \\ & c & \xrightarrow[yc]{} & bc & \end{array} \quad \subseteq \quad H$$

Case 1. If all vertices are mutually distinct: $\Longrightarrow$ H is not discrete corepresentation type.

Case 2. Not vertices are mutually distinct: Use "covering map" of coalgebras and reduce to Case 1.

Definitions:

• A (s-t)-diamond in $C \subseteq \mathbb{K}Q$ is a linear combination of paths starting from s and ending in t.

• A diamond basis of C is a basis containing only diamonds as well as containing all vertices and arrows.

• Any finite dimensional pointed coalgebra has a diamond basis [JMR].

• A covering map $f : C \to D$ is a coalgebra homomorphism, which (1) sends a diamond basis of C to a diamond basis of D; (2) sends diamonds sharing same start vertex or terminal vertex to the same diamond.

• If $f : C \to D$ is a covering map, then $f^* : D^* \to C^*$ is a separable extension of algebras. Hence preserving finite representation types [IS, ISSZ].

Theorem (Iovanov, Sen, Sistko, Zhu)

If H is a connected pointed Hopf algebras of discrete representation type, then the Ext *quiver of H is one of following:*

(0) *A single vertex.*

(1) *A complete oriented cycle;*

(2) $\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots$;

(3)

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & b^2 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdots \\
 & & y^2\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & b & \longrightarrow & ab & \longrightarrow & \cdot & \longrightarrow & \cdots \\
 & & y\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 1 & \xrightarrow{x} & a & \xrightarrow{x^2} & a^2 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

(4) The quiver in (3) identifying vertices $a^m = b^n$. (The quiver looks like a tube.)

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Computing algebra structures for case (3) and (4): $H^{m,n}(\lambda, s, t, k)$ is generated by a, b, x, y satisfying the following conditions, where $\lambda \neq 0, s, t, k \in \mathbb{K}$ ($\mathbb{K}$ algebraically closed, char $\mathbb{K}$=0).

$$ab = ba, a^m = b^n, xy + \lambda yx = k(1 - ab),$$
$$ax + xa = 0, \lambda bx + xb = 0, x^2 = s(1 - a^2),$$
$$by + yb = 0, ay + \lambda ya = 0, y^2 = t(1 - b^2);$$
$$\Delta(a) = a \otimes a, \Delta(b) = b \otimes b,$$
$$\Delta(x) = 1 \otimes x + x \otimes a, \Delta(y) = 1 \otimes y + y \otimes b;$$
$$\epsilon(a) = \epsilon(b) = 1, \epsilon(x) = (y) = 0;$$
$$S(a) = a^{-1}, S(b) = b^{-1}, S(x) = -xa^{-1}, S(y) = -yb^{-1}.$$

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