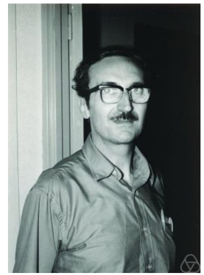
Relative Calabi-Yau completions and higher preprojective algebras

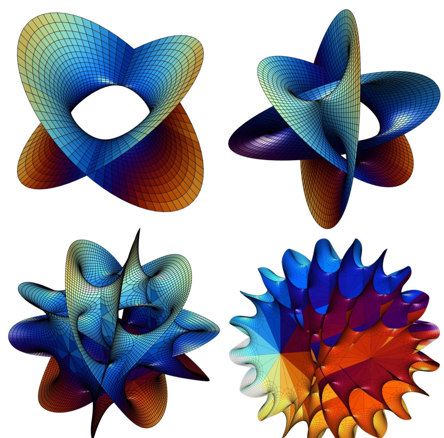
Prehistory

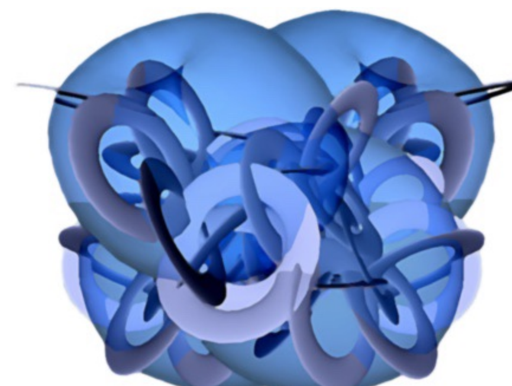
(Absolute) Calabi-Yau varieties



Eugenio Calabi (now 97) conjectured in 1957 that CY-varieties admit Ricci-flat metrics.

Shing-Tung Yau 丘成桐 (now 71) proved Calabi's conjecture in 1977.





Present times

Yilin Wu 吴燚林

Report on part of Yilin Wu's ongoing Ph. D. thesis.

Plan: 0. Brief history and overview in pictures

1. Absolute CY-completions

2. Relative CY-completions

3. A Key equivalence : $S^{-}_{A,B} \sim \tau_n^{-}$

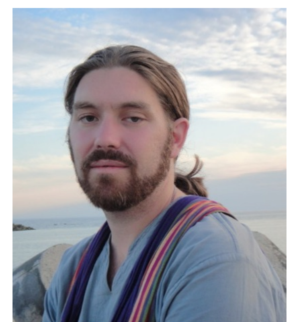
4. Higher preproj. algebras

0. Brief history and overview in pictures

History

- (Right) relative CY-structures were invented by Bertrand Toën in 2014.

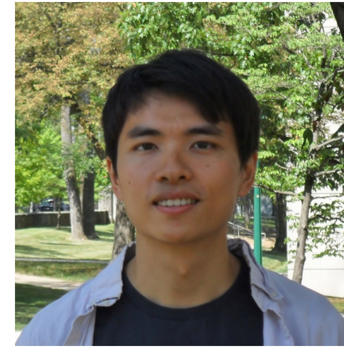
- (Right and left) relative CY-structures were elaborated by Chris Brav and Tobias Dyckerhoff (2016 and 2018). In particular: glueing.



Chris Brav

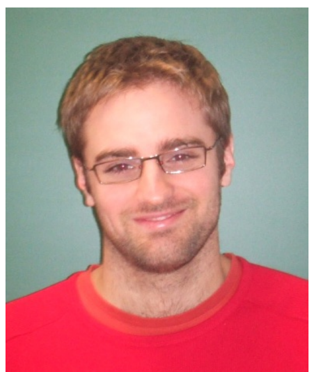
Tobias Dyckerhoff

- Wai-Kit Yeung introduces relative CY-completions in 2016 and advocates the idea that they are non-commutative conormal bundles.

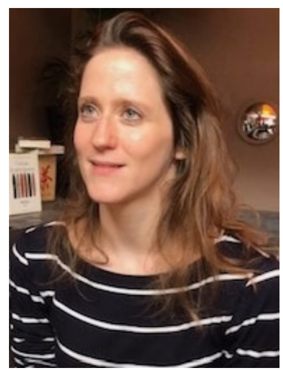


Wai-kit Yeung 楊偉傑 杨伟杰

Tristan Bozec



Damien Calaque

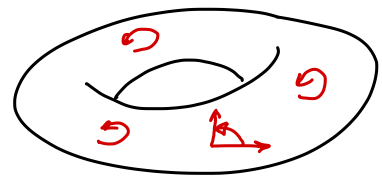


Sarah Scherotzke

- Bozec-Calaque-Scherotzke prove (06/20) that Yeung's idea is justified by Kontsevich-Rosenberg's criterion.

Overview in pictures

dg algebra
+
CY-structure
(absolute)



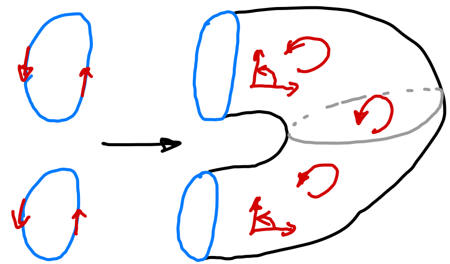
manifold
+
orientation

dg algebra morphism

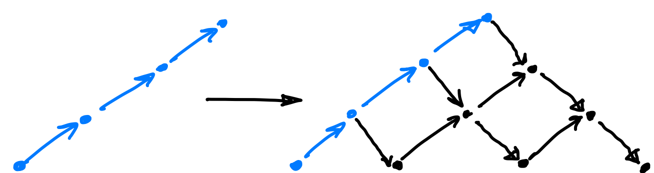
$B \to A$
+
relative CY-structure

BOUNDARY

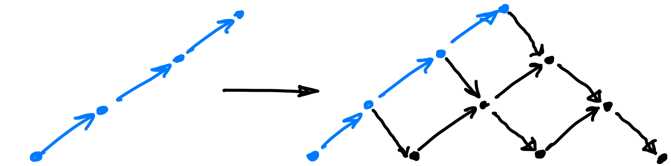
AUSLANDER



manifold with boundary
+
orientation

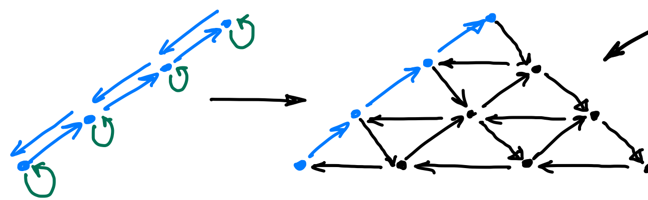


NO relative CY-structure here!



NO relative CY-structure here!

relative | 3-CY completion



Geiss-Leclerc-Schröer 2007

2-dim. (abs.) Ginzburg alg.

H^0: preproj. alg.

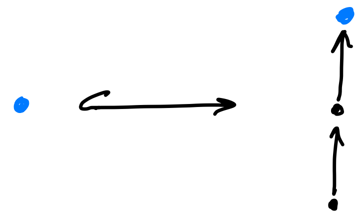
$\dim H^* = \infty$

3-dim. relative Ginzburg alg.

$W = \sum \triangle - \sum \triangledown$

surprise! $\begin{cases} H^* = H^0 : & \partial_\alpha W = 0, \ \alpha \text{ not frozen} \\ \dim H^0 < \infty & \end{cases}$

Exercise: Compute the rel. 2-CY completion of



Hint: The 2-dim. rel. Ginzburg algebra is
the Auslander algebra of a 4-dim. selfinj. algebra.

1. Absolute CY-completions (2011)

k a perfect field (for simplicity)

A a dg (= differential graded) algebra (assoc., with 1, non com.)

$\mathcal{C}A = \{$ dg right A-modules$\}$ ($A = A^0$: complexes of right A-mod.)

$\mathcal{D}A$ = derived category $= (\mathcal{C}A)[\mathrm{Qis}^{-1}]$: triang. with arb. coprod.

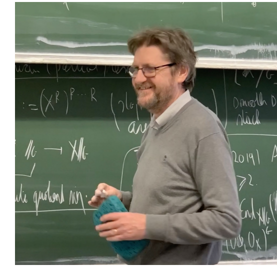
$\mathrm{per}\,A = \{ C \in \mathcal{D}A \mid C \text{ compact} \} = \mathrm{thick}(A_A)$ ($A = A^0$: $\mathcal{K}^b(\mathrm{proj}A)$)

$A^e = A \otimes A^{op}$: $\mathcal{C}A^e = \{$dg A-A-bimodules$\}$

A is smooth $\Longleftrightarrow$ ${}_AA_A \in \mathrm{per}(A^e)$

$M \in \mathcal{D}A^e$: $M^\vee = \mathrm{RHom}_{A^e}(M, A^e) \in \mathcal{D}((A^e)^{op}) \simeq \mathcal{D}(A^e)$

Ω_A = inverse dualizing complex = $A^\vee$



Fix $n \in \mathbb{Z}$.

Def. (Ginzburg '06, VdB '11): A is n-CY $\Longleftrightarrow$ $A^\vee \simeq \Sigma^{-n} A$ in $\mathcal{D}A^e$.

Suppose B is a smooth dg algebra.

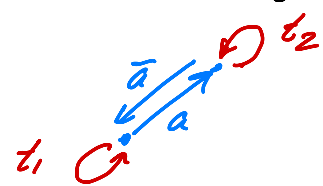
Def. ('11): $\Pi_{n+1} B$ = (abs.) $(n+1)$-CY-completion

$$= T_B(\omega) = B \oplus \omega \oplus (\omega \otimes_B \omega) \oplus \dots, \quad \omega = \text{cofibrant res. of } \Sigma^n \Omega_A$$

Thm ('11): $\Pi_{n+1} B$ is smooth and canonically $(n+1)$-CY.

Example: B: 

$\Longrightarrow$ $\Pi_2 B$ = 2-dim. Ginzburg algebra:



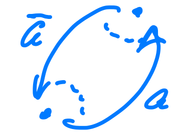
$|t_i| = -1$

$dt_1 = -\bar{a}a$

$dt_2 = a\bar{a}$.

$$\Longrightarrow \dim H^* \Pi_2 B = \infty \quad (\Longleftarrow Q_1 \neq \emptyset)$$

$$\Longrightarrow H^0(\Pi_2 B) = \text{preproj. alg.} :$$



2. Relative CY-completions

$B \longrightarrow A$ morphism of dg algebras, A, B smooth.

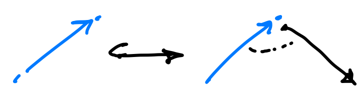
Def.: ① $\Omega_{A,B}$ = relative inverse dualizing complex $= (\mathrm{cone}(A \overset{L}{\underset{B}{\otimes}} A \longrightarrow A))^{\vee} \in \mathcal{D}A^e$
(Yeung '16)

② $\Pi_{n+2}(A,B)$ = rel. $(n+2)$-dim. Ginzburg alg. of $B \longrightarrow A$

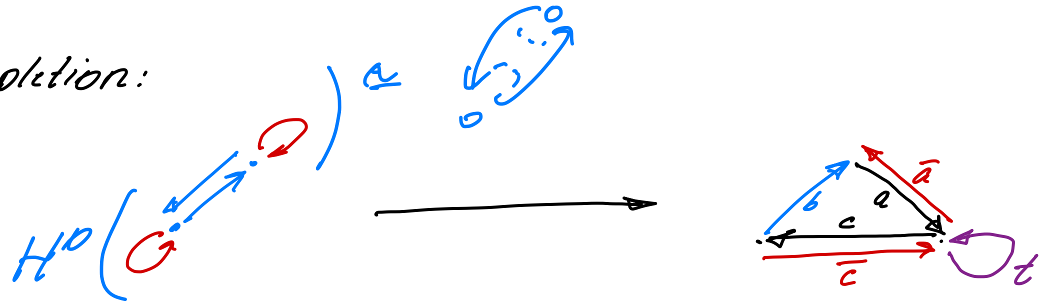
$$= T_A(\omega), \quad \omega = \text{cofib. res. of } \Sigma^{n+1}\Omega_{A,B}.$$

③ (Rel. $(n+2)$-CY completion of $B \to A$) $= (\Pi_{n+1} B \xrightarrow[\text{can}]{} \Pi_{n+2}(A,B))$

Example: $n=1$: $(B \longrightarrow A) =$



Rel. 3-CY-completion:



abs. 2-CY completion

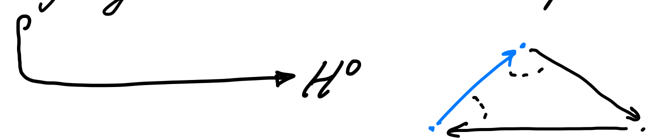
= abs. 2-dim. Ginzburg alg.

rel. 3-dim. Ginzb. alg.:

$d(\bar a) = bc$, $d(\bar c) = ab$,
$d(t) = a\bar a - \bar c c$.

$\dim H^* = \infty$, H^0 = preproj. alg.

$H^* = H^0$ is fin. dim. (!)



Thm (Y16, BCS20): $(\Pi_{n+1} B \to \Pi_{n+2}(B,A))$ is smooth and can. rel. $(n+2)$-CY.

Not.: $S^-_{A,B}$ = "rel. inverse Serre functor"

$$= ? \overset{\mathbb{L}}{\otimes}_A \Omega_{A,B} : \mathcal{D}^b A \longrightarrow \mathcal{D}^b A$$

Rk: Thus, $\Pi_{n+2}(A,B) \simeq \bigoplus_{i\geq 0} (\Sigma^{n+1} S^-_{A,B})^i (A)$ in $\mathcal{D}^b A$.

3. A key equivalence: $S^-_{A,B} \sim \tau_n^-$

$n \geq 1$ an integer

B a fin. dim. algebra. Suppose B is n-repres. finite [IO11], i.e.

a) mod B contains an n-cluster tilting object M, i.e. $\mathcal{M} = \operatorname{add} M$ satisfies

$$\mathcal{M} = \{ X \in \operatorname{mod} B \mid \operatorname{Ext}_B^{0<i<n}(X, \mathcal{M}) = 0 \} = \{ X \in \operatorname{mod} B \mid \operatorname{Ext}^{0<i<n}(\mathcal{M}, X) = 0 \}.$$

b) $\operatorname{gldim} B \leq n$.

Let $S = ? \overset{\mathbb{L}}{\underset{B}{\otimes}} DB : \mathcal{D}^b B \longrightarrow \mathcal{D}^b B$ be the Serre functor and define

$$\begin{tikzcd} \operatorname{mod} B \arrow[r, hook, "\mathrm{can}"'] \arrow[rrr, bend right, "\tau_n^-"'] & \mathcal{D}^b B \arrow[r, "\Sigma^n S^{-1}"'] & \mathcal{D}^b B \arrow[r, "H^0"'] & \operatorname{mod} B \end{tikzcd}$$

$\tau_n^- :=$ higher inverse AR-translation (Iyama '07)

We have $\mathcal{M} = \operatorname{add}\{\tau_n^{-i} B \mid i \geq 0\}$ [I07].

Put $A = (n\text{-Auslander alg. of } B) = \operatorname{End}_B(M)$.

Abuse of notation:

$$\begin{tikzcd} \operatorname{add} M \arrow[r, "\sim"] \arrow[d, "\tau_n^-"'] \arrow[dr, phantom, "\cong"] & \operatorname{proj} A \arrow[r, hook] \arrow[d, "\tau_{n,B}^-"] \arrow[dr, phantom, "\cong"] & \operatorname{mod} A \arrow[d, "\tau_{n,B}^-"] \\ \operatorname{add} M \arrow[r, "\sim"] & \operatorname{proj} A \arrow[r, hook] & \operatorname{mod} A \end{tikzcd}$$

right ex. extension

KEY LEMMA: Up to shift and derivation $S^-_{A,B} : \mathcal{D}^b A \to \mathcal{D}^b A$ is $\tau^-_{n,B}$!

More precisely, as functors $\mathcal{D}^b A \to \mathcal{D}^b A$:

$$\boxed{\Sigma^{n+1} S^-_{A,B} \simeq \mathbb{L}\, \tau^-_{n,B}}$$

Consequence:

$$\Pi_{n+2}(A,B) \underset{\text{in } \mathcal{D}^b A}{\simeq} \bigoplus_{i \geq 0} (\Sigma^{n+1} S^-_{A,B})^i (A)$$

$$\simeq \bigoplus_{i \geq 0} \underbrace{\tau^{-i}_{n,B}(A)}_{\in \operatorname{proj} A\,!}$$

Cor.: a) $\Pi_{n+2}(A,B)$ is conc. in deg. 0, fin. dim., of finite gldim. and the functor $H^0 \Pi_{n+1} B \to \Pi_{n+2}(A,B)$ is fully faithful

b) $\Pi_{n+2}(A,B)$ is internally bimodule $(n+2)$-CY w.r.t. supp(B) in the sense of Pressland '17.

Rk: Using the Cor. and the fact that $H^0 \Pi_{n+1} B = (n+1)$-preproj. alg. $= \tilde{B}$ is selfinjective [IO13], we obtain a new proof of the fact that $\underline{\operatorname{mod}} \tilde{B}$ contains an $(n+1)$-cluster tilting object (first proved in [IO13]).

The proof consists in showing the equivalence between the following two diagrams:

$$
\begin{tikzcd}
 & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[d] & \mathcal{H}^b(\mathcal{P}) \arrow[d] \\
\mathcal{S} \arrow[r, hook] \arrow[d, equal] & \operatorname{per}\Gamma \arrow[r] \arrow[d] & \mathcal{C}^{rel}_{\Gamma} \arrow[d] \\
\mathcal{S} \arrow[r, hook] & \operatorname{per}\bar{\Gamma} \arrow[r] & \mathcal{C}_{\bar{\Gamma}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{per}\Gamma \supseteq \mathcal{F} \arrow[r, "\sim"] \arrow[d] & \mathcal{F}' \subseteq \mathcal{C}^{rel}_{\Gamma} \arrow[d] \\
\operatorname{per}\bar{\Gamma} \supseteq \bar{\mathcal{F}} \arrow[r, "\sim"] & \mathcal{C}_{\bar{\Gamma}}
\end{tikzcd}
$$

[Palu '09]

$$
\begin{tikzcd}
 & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[d] & \mathcal{H}^b(\mathcal{P}) \arrow[d] \\
\mathcal{H}^b_{ac}(\mathcal{M}) \arrow[r] \arrow[d, equal] & \mathcal{H}^b(\mathcal{M}) \arrow[r] \arrow[d] & \mathcal{D}^b(\mathcal{E}) \arrow[d] \\
\mathcal{H}^b_{ac}(\mathcal{M}) \arrow[r] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r] & \underline{\mathcal{E}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}^b(\mathcal{M}) \supseteq \mathcal{F}'' \arrow[r, "\sim"] & \mathcal{E} \subseteq \mathcal{D}^b(\mathcal{E}) \arrow[d] \\
 & \underline{\mathcal{E}}
\end{tikzcd}
$$

$\Gamma = \Pi_{n+2}(A,B)$, $\bar{\Gamma} = \Pi_{n+2}(A/\langle B \rangle)$, $\mathcal{S} = \operatorname{thick}(S_i \mid i \notin \operatorname{supp} B)$

$\mathcal{E} = \operatorname{mod}\tilde{B}$, $\mathcal{P} = \operatorname{proj}\tilde{B}$,

$\mathcal{H}^b_{ac}(\mathcal{M}) = \{X \in \mathcal{H}^b(\mathcal{M}) \mid X \text{ acyclic in } \mathcal{E}\}$

$\mathcal{F}$ = rel. fund. domain $\subseteq \operatorname{per}\Gamma$

$= \{ X \in \operatorname{per}\Gamma \mid X = (\ldots 0 \to X_n \longrightarrow \ldots \longrightarrow X_0 \to 0 \to \ldots),$

$H_i \operatorname{Hom}(X_\bullet, F) = 0,\ \forall 1 \leq i \leq n,\ \forall F \in \operatorname{add}\Gamma,\ F$ "frozen" $\}$

$\mathcal{F}'' = \{ X \in \mathcal{H}^b(M) \mid$ similarly with $F \in \mathcal{P} \}$

Rks: 1) The algebra $\Gamma = \Pi_{n+2}(A,B)$ is isom. to the end. alg. of the can. $(n+1)$-cto in $\mathcal{E} = \operatorname{mod}\tilde{B}$.

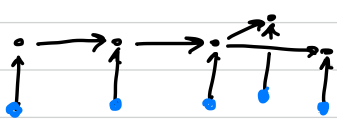
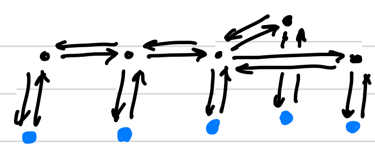
Γ does not appear in [IO13]. The algebra $\overline{\Gamma}$ is isom. to the corresp. stable end. alg..

2) The cat. $\mathcal{U} = \{ (\Sigma^n S^{-1})^i (B) \mid i \in \mathbb{Z} \} \subseteq \mathcal{D}^b B$ of [IO13] does not appear in our setup.

3) The contents of p. 13 adapt well to the setup of n-complete algebras [I11]

(and probably many other settings of higher AR-theory).

Appendix: Ubiquity of 2-dim rel. Ginzburg algebras

Morphism	2-dim. rel. G. alg.	
		Auslander algebra of $k[x]/(x^3)$.
		Auslander alg. of the Bass order[1)] $B_3 = \begin{bmatrix} R & Rx^3 \\ R & R \end{bmatrix}$, $R = k[[x]]$ with indec. $[R \;\; Rx^i]$, $0 \le i \le n$
		Nakajima algebra of type D_5

1) We are very grateful to Osamu Iyama for this class of examples.