GKZ systems and perverse schobers

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1. RH correspondence
2. GKZ systems
3. perverse sheaves on hyperplane arrangements
4. perverse schobers
5. categorification of GKZ systems

1. Riemann - Hilbert correspondence

system of lin. part. diff. equations $\longleftrightarrow$ holonomic $\mathcal{D}$-module + reg. sing. ? $\longrightarrow$ constructible coh. solutions

e.g. $P_1\Phi = \dots = P_m\Phi = 0$ $\qquad M = \mathcal{D}/(\sum \mathcal{D} P_i)$ $\xrightarrow{Sol}$ $R\mathrm{Hom}_{\mathcal{D}}(M, \mathcal{O})$

Ex. $x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$
(Euler) Gauss hypergeometric eq.

$\mathcal{D} = \mathbb{C}\langle x, \partial_x \rangle$
$M = \mathcal{D}/\mathcal{D}(x(1-x)\partial_x^2 + (c-(a+b+1)x)\partial_x - ab)$

- $y_1 = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$ around 0
- $y_2 = \dots$

$$D^b_{rh}(\mathrm{mod}\, \mathcal{D}_X) \xrightarrow{\sim} D^b_c(\mathbb{C}_X)$$

$\mathrm{mod}\, \mathcal{D}_X \longmapsto \mathrm{Perv}(X)[-\dim X]$

perverse sheaves

constructible sheaf $\mathcal{F}$
- $\dim \mathcal{F}_x < \infty$
- $\exists\, X = \bigsqcup X_i$. $\mathcal{F}|_{X_i}$ loc. const.

2. GKZ systems [Gelfand, Kapranov, Zelevinsky '89]

generalisation of classical hypergeometric dfgl. eq. (Gauss, Horn, Appell ..)

- $X = \mathbb{C}^{*d} \,\big|\, A \subset \mathbb{Z}^{d-n},\ |A| = d$, $\alpha \in \mathbb{C}^{d-n}$ s.t. $\mathbb{Z}A = \mathbb{Z}^{d-n}$, $\exists h \in (\mathbb{Z}^{d-n})^\vee . \langle h, a\rangle = 1\ \forall a \in A$

- $Z_i = \sum_j a_{ji}\, x_j \partial_j$

  $Z_i - \alpha_i$ homogeneity relations

$(Z_i - \alpha_i)\Phi = 0,\ i = 1,\dots,d-n \iff \forall t \in \mathbb{C}^{*d-n}$

$\Phi(t^{a_1} x_1, \dots, t^{a_{d-n}} x_d) = t^{\alpha} \Phi(x_1, \dots, x_d)$

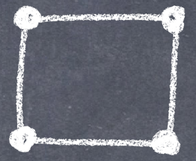
$$\Box_l = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i}, \quad l \in L = \ker A \ (A : \mathbb{Z}^d \to \mathbb{Z}^{d-n})$$

- GKZ system (of parameter α)

$$\mathcal{P}(\alpha) := D_x \Big/ \sum_{l \in L} D_x \Box_l + \sum_{i=1}^{d-n} D_x (Z_i - \alpha_i)$$

Example (Gauss hypergeometric eq.)

$X = \mathbb{C}^{*4}$, $A = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad h = (-1, 1, 1)$



$L = \mathbb{Z} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad \alpha = \begin{pmatrix} c-1 \\ -a \\ -b \end{pmatrix}$

$$(-x_1 \partial_1 + x_2 \partial_2)\Phi = (c-1)\Phi$$

$$(x_1 \partial_1 + x_3 \partial_3)\Phi = -a\Phi$$

$$(x_1 \partial_1 + x_4 \partial_4)\Phi = -b\Phi$$

$$(\partial_1 \partial_2 - \partial_3 \partial_4)\Phi = 0$$

$\Longrightarrow$ Gauss hypergeometric eq.

$$\left(x_3^{-1} x_4^{-1} \left(x_1 \partial_1^2 - (1+a+b) x_1 \partial_1 - ab \right) - x_2^{-1} \left(x_1 \partial_1^2 - c \partial_1 \right) \right) \Phi = 0$$

$F(x) := \Phi(x, 1, 1, 1)$ sol. of Gauss h.e.

(by homogeneity F determines Φ)

GKZ system - descent + properties

$$0 \to \mathbb{Z}^n \xrightarrow{B^*} \mathbb{Z}^d \xrightarrow{A} \mathbb{Z}^{d-n} \to 0 \quad / \otimes \mathbb{C}^*$$

$$0 \to T \xrightarrow{B^*} \mathbb{T} \xrightarrow{A} H \to 0 \quad \Longrightarrow \quad 0 \to X(H) \xrightarrow{A^*} X(\mathbb{T}) \xrightarrow{B} X(T) \to 0$$

$X = \mathbb{T}^*$

GKZ system $\overset{\text{weakly}}{H^*}$-invariant $\Longrightarrow$ descents to $\mathbb{T}^*/_{H^*} = T^* \to D_{T^*}$-mod. $\overline{\mathcal{J}(\alpha)}$

[GKZ, Adolphson '94] $\mathcal{J}(\alpha)$ is holonomic, with reg. sing.

$\overline{\mathcal{J}(\alpha)}$ — " —

$P(\alpha) := Sol(\overline{\mathcal{J}(\alpha)})[d]$

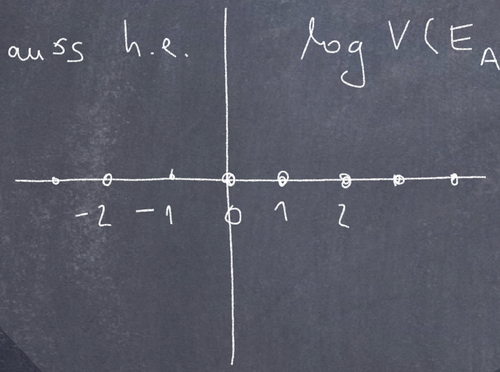
[GKZ '89] If α is "non-resonant" then $P(\alpha)$ is simple.

singular locus $V(E_A)$, E_A GKZ discriminant

for Gauss h.e. — $V(E_A) = \{1\}$

— in general complicated, sometimes hyperplane arrangement

[Kite '17] If B is quasi-symmetric, then $V(E_A)$ is hyperplane arrangement (in log. coordinates)



quasi-symmetry

$b_i := Be_i \in \mathbb{Z}^n \subseteq \mathbb{R}^n$

B is q.s. if $\forall$ vector line $\ell \in \mathbb{R}^n$. $\sum_{b_i \in \ell} b_i = 0$

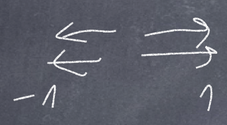
log. coordinates

$\mathbb{C}^n \cong X(T)_{\mathbb{C}} = \mathrm{Lie}(T^*) \xrightarrow{e^{2\pi i(-)}} T^*$

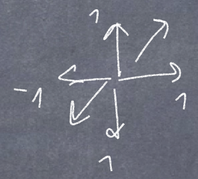
B quasi-symmetric

B q.s.

$\Delta = \frac{1}{2}$ (zonotope spanned by b_i) $\subseteq X(T)_{\mathbb{R}}$



$\Delta = [-1, 1]$



$\Delta =$



$\mathcal{H}$ = (supp. hyperplanes of Δ) + $X(T)$

$\mathcal{H}_{\mathbb{C}} \xrightarrow{e^{2\pi i(-)}} V(E_A)$ (up to shift on LHS)

$(\mathcal{L}, \leq)$ stratification of $X(T)_{\mathbb{R}}$ induced by $\mathcal{H}$

Solutions of GKZ system, α non-resonant

Mellin Barnes integral

$$M(x_1, \dots, x_d) = \int_{\zeta + i\mathbb{C}^n} \prod_{j=1}^{d} (-\gamma_j - \langle b_j, s \rangle) x_j^{\gamma_j + \langle b_j, s \rangle} ds$$

$\gamma \in \mathbb{C}^d$. $A\gamma = \alpha$

M multi valued function on a subset of T^*

— " —

single-valued — " — $T^*/X(T)_{\mathbb{C}}$

$$\Pi \dashrightarrow T^* \to D$$

$M_\chi(x) := M(x - \chi)$, $\chi \in X(T)$, $C \in \mathcal{C}$ chamber, $\nu_C \in C$

[Beukers '16] $\{M_\chi \mid \chi \in (\nu_C + \Delta) \cap X(T)\}$ is a basis of sol.

on $C \times i X(T)_{\mathbb{R}}$ of the GKZ system with par. α

3. Perverse sheaves on hyperplane arrangements

$\mathcal{H}$ affine real hyperplane arrangement $\subset V$

$(\mathcal{C}, \leq)$ stratification of V induced by $\mathcal{H}$

$\mathrm{Perv}(V_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}})$? ← quiver description

$V = \mathbb{R}$ $\quad \mathcal{H} = \{0\}$

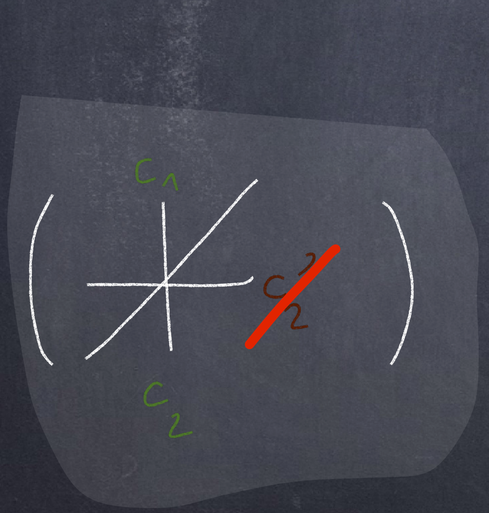
$$\mathrm{Perv}(\mathcal{C}, \mathcal{H}_{\mathbb{C}}) \cong \left\{ E_- \underset{\gamma_-}{\overset{\delta_-}{\rightleftarrows}} E_0 \underset{\gamma_+}{\overset{\delta_+}{\rightleftarrows}} E_+ \;\middle|\; \gamma_\mp \delta_\mp = \mathrm{id}_{E_\mp},\ \gamma_\mp \delta_\pm : E_\pm \xrightarrow{\sim} E_\mp \right\}$$

[Kapranov, Schechtman '16]

Perv $(V_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}}) \cong (E_c, \delta_{cc'}, \gamma_{c'c})_{c,c' \in \mathcal{C},\, c' \leq c}$

- $E_{c'} \xleftarrow{\delta_{cc'}} E_c$ rep. of $(\mathcal{C}, \geq)$ in $\mathrm{Vect}(\mathbb{C})$
- $E_{c'} \xrightarrow{\gamma_{c'c}} E_c$ rep. of $(\mathcal{C}, \leq)$ in $\mathrm{Vect}(\mathbb{C})$

(m) $\gamma_{c'c}\delta_{cc'} = \mathrm{id}_{E_c}$ $\left(\Rightarrow \phi_{c_1c_2} = \gamma_{c'c_2}\delta_{c_1c'} \text{ for } c' \leq c_1, c_2\right)$



(i) $\phi_{c_1c_2}$ iso. $\forall c_1, c_2$, $\dim c_1 = \dim c_2 = \dim c_1 \wedge c_2 + 1 =: d$
c_1, c_2 lie in the same d-dim. aff. sp.

(+) $\phi_{c_1c_3} = \phi_{c_2c_3}\phi_{c_1c_2}$ $\forall$ collinear triple c_1, c_2, c_3
$(\exists a_i \in c_i \,.\, a_2 \in [a_1, a_3])$

4. Perverse Schobers

MS : X cy , K_X = stringy Kähler moduli space of X
("parametrizes complex structures on mirror dual X° of X")

$\pi_1(K_X) \curvearrowright D^b(X)$ "local system of triangulated cat. on K_X with fiber $D^b(X)$"

loc. sys. of Δ-cat $\longrightarrow$ pers. sh. of Δ-cat

[Kapranov, Schechtman '15] perverse schobers
("coefficents in Fukaya cat.")

[Bondal, KS '16] Perverse schobers on hyp. arr.

$\mathrm{Perv}(V_{\mathcal{C}}, H_{\mathcal{C}}) \cong (\mathcal{E}_c, S_{cc'}, \gamma_{c'c})_{c,c' \in \mathcal{C},\, c' \le c}$

(S, γ) adjoint pair

- $\mathcal{E}_{c'} \xleftarrow{S_{cc'}} \mathcal{E}_c$ rep. of $(\mathcal{C}, \ge)$ in Cat_Δ
- $\mathcal{E}_{c'} \xrightarrow{\gamma_{c'c}} \mathcal{E}_c$ rep. of $(\mathcal{C}, \le)$ in Cat_Δ

(m) $\gamma_{c'c} S_{cc'} \cong \mathrm{id}_{\mathcal{E}_c}$ $\left(\Rightarrow \phi_{c_1 c_2} = \gamma_{c' c_2} S_{c_1 c'} \text{ for } c' \le c_1, c_2\right)$

(i) $\phi_{c_1 c_2}$ eq. $\forall c_1, c_2$, $\dim c_1 = \dim c_2 = \dim c_1 \wedge c_2 + 1 =: d$
c_1, c_2 lie in the same d-dim aff. sp.

(+) $\phi_{c_1 c_3} \cong \phi_{c_2 c_3} \phi_{c_1 c_2}$ $\forall$ collinear triple c_1, c_2, c_3
$(\exists a_i \in c_i \cdot\ a_2 \in [a_1, a_3])$

5. Categorification of GKZ system

$W = \mathbb{C}^i \circlearrowright T \supset T$ (T-weights of W are b_i)

$X(T) \xrightarrow{B} X(T) \to 0$

[Halpern-Leistner, Sam '16, Kite '17] $SKMS(W,T) = X(T)_{\mathbb{C}} \setminus \log V(E_A)$
$= X(T)_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$

[H-L S] $\pi_1(X(T)_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}) \curvearrowright D^b(W^{ss,\chi}/T)$
(GIT quotient)

Aim: Extend this loc. system to a perverse schober $\mathcal{G}$ s.t. $K^\circ(\mathcal{G})$ is GKZ perv. sheaf

[Donovan '18] $W = \mathbb{C}^4$, $B = (-1,-1,1,1)$ perv. schober

$\mathbb{T} \cong T \times H$, $\quad X(\mathbb{T}) \xrightarrow{B} X(T) \to \mathcal{O}$

$\mathcal{Y}_c = \langle \chi \otimes \mathcal{O}_W \mid B\chi \in (-\nu_c + \Delta) \cap X(T) \rangle \subset D^b(W/\mathbb{T})$, $\nu \in C$

$\mathcal{Y}_{c'} \hookleftarrow \mathcal{Y}_c$

$\mathcal{Y}_c \hookrightarrow \mathcal{Y}_c$

$\uparrow$

right adjoint to $j_{cc'}$ (exists!)

($\Delta = [-1,1]$ — $\overset{c'}{\underset{-1}{\bullet}}$ — $\overset{c}{\underset{0}{\bullet}}$ — $\underset{1}{\bullet}$ — , $\quad \nu_c = \frac{1}{2}$, $\nu_{c'} = 0$

$(-\nu_c + \Delta) \cap X(T) = [-\frac{3}{2}, \frac{1}{2}] \cap X(T) = \{-1, 0\}$

$(-\nu_{c'} + \Delta) \cap X(T) = [-1, 1] \cap X(T) = \{-1, 0, 1\}$)

$\mathcal{Y}_c \circlearrowright H \quad \Rightarrow K^\circ(\mathcal{Y}_c)_c \quad \mathbb{C}[H]$-module

$S(h) := \left(K^\circ(\mathcal{Y}_c) \otimes_{\mathbb{C}[H]} \mathbb{C}_h \, , \, K^\circ(j_{cc'}), K^\circ(\gamma_{c'c}) \right)_{c', c \in \mathcal{C}, \, c' \leq c}$

[Š, VdB '19] $(\mathcal{Y}_C, \delta_{CC'}, \gamma_{C'C})_{C,C' \in \mathcal{L},\, C' \leq C}$ is $X(T)$-equivariant perverse schober on $X(T)_{\mathbb{C}}$.

[Š, VdB '20] $\alpha \in \mathbb{C}^{d-m}$ $(\cong \mathrm{Lie}(H))$ non-resonant

$$P(\alpha) \cong S(e^{2\pi i \alpha})$$

Pf. M_χ (nB integral) $\longmapsto [(-\chi) \otimes \mathcal{O}_W]$

- determine monodromy of GKZ system
- identify with $\phi_{C_1 C_2}$
- local systems are $\cong$
- $S(e^{2\pi i \alpha})$ is simple
- $\cong$ of perv. sheaves

Thamk you!