Bounded extension algebras and Han's conjecture

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joint with Claude Cibils, Marcelo Lanzilotta and Eduardo Marcos

Some introductory comments

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Conjecture

(We will say that A is smooth *if $gldim(A) < \infty$).*

$$A \textit{ is } \text{smooth} \iff \exists n |\, \mathrm{HH}_i(A) = 0\ \forall i > n.$$

Remark

The right hand condition actually implies that $\mathrm{HH}_i(A) = 0$ for all $i > 0$, using a result by Keller, relating Hochschild and cyclic homology.

Non exhaustive list of cases for which the conjecture is known to be true:

- A commutative, not necessarily f.d. but finitely generated (Avramov-Vigué, BACH).
- A monomial (Han).
- $(char(k) = 0)$, A Koszul $+$ other families of local graded algebras (Bergh-Madsen) and more recently, without hypothesis on $char(k)$ for trivial extensions of graded algebras.
- Quantum complete intersections (Bergh-Erdmann).
- $k\langle x_1, \ldots, x_n\rangle/(f_1, \ldots, f_p)$ non necessarily fin. dim. with $f_1 \in k[x_1], f_i \in (x_2, \ldots, x_n), \forall i \geq 2$ (S-Vigué).
- GWA (S., Suárez Alvarez, Vivas).

Commutative case

In this case, $\mathrm{HH}_n(A) = \bigoplus_{i=1}^n \mathrm{HH}_n^{(i)}(A)$ such that $\mathrm{HH}_n^{(n)}(A) = \Lambda^n \Omega^1_{A/k}$ (exterior power of Kähler differentials) and

$$\mathrm{HH}_n^{(1)}(A) = D_{n-1}(A/k) \cong Harr_n(A)$$

(where $D_{n-1}(A/k)$ is André-Quillen homology and $Harr_n(A)$ is Harrison homology).

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André-Quillen homology satisfies a Jacobi-Zariski long exact sequence, which can be considered a "change of rings" sequence. Its annihilation is closely related to the smoothness of A. In fact, if A is smooth, $\mathrm{HH}_n(A) = \mathrm{HH}_n^{(n)}(A)$ (HKR).

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In the non commutative setting A-Q homology does not exist but Kaygun has obtained a J-Z sequence for any extension of k-algebras $B \subset A$ such that A is B-flat. This is a quite restrictive hypothesis which made us think about possible generalizations.

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We recover a Jacobi-Zariski long exact sequence involving Hochschild homology and relative Hochschild homology.

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In the first part I will explain our main result to appear in Pacific J. Math. where we prove that the class $\mathcal{H}$ of finite dimensional algebras which verify Han's conjecture is closed under split bounded extensions (to be defined later). More precisely if $A = B \oplus M$ is such an extension, then

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In the second part I will talk about a generalization of this result for A non necessarily split.

Split extensions

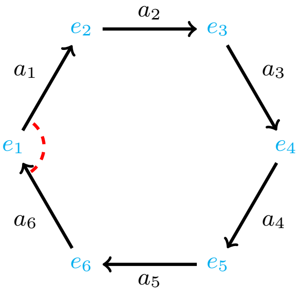
We consider algebras of type $A = B \oplus M$, where B is a subalgebra and M a two-sided ideal and we prove that under certain hypotheses on M, the algebra A satisfies Han's conjecture if and only if B does.

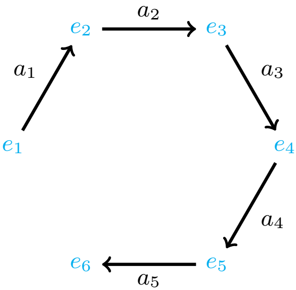
Our tools:

- relative homology and reduced bar resolution.
- Jacobi-Zariski long exact sequence if:
  - $\exists n \in \mathbb{N}$ such that $M^{\otimes_B n} = 0$ -in this case we will say that M is B-tensor nilpotent-,
  - $pdim_B M_B < \infty$,
  - either ${}_B M$ or M_B is projective.

  If these conditions hold we will say that $A = B \oplus M$ is a *bounded* split extension.

Example





$A = kQ/\langle R \rangle$ with $R = \{a_1 a_6\}$, $B = kQ'$.

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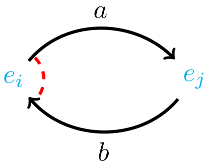

$A = kQ/\langle R\rangle$ with $R = \{ab\}$. We can choose:

1. $B_1 = kQ_0 \oplus k \cdot a$ and $M_1 = \langle b\rangle$ or
2. $B_2 = kQ_0 \oplus k \cdot b$ and $M_2 = \langle a\rangle$

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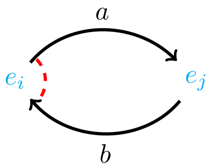

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3. $B = kQ_0$ and $M = \langle a, b\rangle$ because it is not B-tensor nilpotent.

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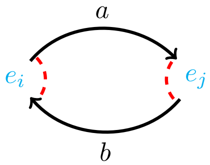

$A = kQ/\langle R\rangle$ with $R = \{ba, ab\}$. Here we cannot choose $B = kQ_0$ and $M = \langle a, b\rangle$ because it is not B-tensor nilpotent, and we can neither choose $M_1 = \langle b\rangle$ nor $M_2 = \langle a\rangle$ because it is neither left nor right B-projective.

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- relative projective resolutions:

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to X \to 0$$

where each P_i is a relative projective A-module, the d's are A-morphisms, $d^2 = 0$ and there exists a B-contracting homotopy, so $\mathrm{Tor}_*^{A|B}(X, Y)$ is well defined.

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Definition

The Hochschild homology of A relative to B with coefficients in an A-bimodule X is $\mathrm{H}_*(A|B, X) = \mathrm{Tor}_*^{A^e|B^e}(X, A)$.

Recall: $B \subset A$ is split if $\exists \pi : A \to B$ morphism of algebras such that $\pi(b) = b, \forall b \in B$.

Examples

- *Given an algebra B and a B-bimodule N, consider $T = T_B(N) = B \oplus \ N \ \oplus \ N \otimes_B N \ \oplus \cdots$. Then $T = B \oplus T^{>0}$, is a split extension. Moreover, if $J \subset T^{>0}$ is a two-sided ideal of T, then $B \subset T/J$ is a split extension as well.*
- *$B = kQ_0 \subset kQ/I$ with I admissible.*
- *$B = kQ/I$ bound quiver algebra, F finite set of new arrows with two maps $s, t : F \to Q_0$. Let Q_F be the quiver such that $(Q_F)_0 = Q_0$ and $(Q_F)_1 = Q_1 \sqcup F$.*
  If $B_F = kQ_F/\langle I \rangle_{kQ_F}$, then $B_F = T_B(N)$ where

  $$N = \bigoplus_{a \in F} Bt(a) \otimes s(a)B.$$

  Let $J \subset B_F^{>0}$ be a two-sided ideal of B_F. The algebra

  $$A = B_F/J = B \oplus (B_F^{>0}/J)$$

  is also a split extension.

The reduced resolution of a split extension

Theorem

Let $A = B \oplus M$ be a split extension of algebras. There is a reduced relative bar resolution of A as A-bimodule

$$\cdots \xrightarrow{d} A \otimes_B M^{\otimes_B n} \otimes_B A \xrightarrow{d} \cdots \xrightarrow{d} A \otimes_B M \otimes_B A \xrightarrow{d} A \otimes_B A \xrightarrow{d} A \to 0$$

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Idea of proof contracting homotopy t (left B-linear and right A-linear),

$t(a_1 \otimes m_2 \otimes \cdots \otimes m_{n+1} \otimes a_{n+2}) = 1 \otimes (a_1)_M \otimes m_2 \otimes \cdots \otimes m_{n+1} \otimes a_{n+2}$.

Corollary

If M is B-tensor nilpotent and X is an A-bimodule, then in large enough degrees

$$\mathrm{H}_*(A|B, X) = 0 \quad \textit{and} \quad \mathrm{H}^*(A|B, X) = 0.$$

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This happens for example when $A = kQ_0 \oplus M$ and Q does not contain oriented cycles.

Relation between relative and usual Hochschild homology

Proposition

Let $A = B \oplus M$ be a split extension of algebras, and let X be an A-bimodule. For $* \geq 1$, there is a sequence of chain complexes

$$0 \to C_*(B, X) \xrightarrow{\iota} C_*(A, X) \xrightarrow{\kappa} C_*^M(A|B, X) \to 0$$

where ι is injective, κ is surjective and $\kappa\iota = 0$.
In degree 0 we have the sequence

$$0 \to X \xrightarrow{1} X \to X_B \to 0.$$

This led us to consider **nearly exact sequences**.

Nearly exact sequences

Definition

A sequence of chain complexes concentrated in non negative degrees

$$0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0$$

is nearly exact *if:*

1. ι *is injective,*
2. κ *is surjective,*
3. $\kappa\iota = 0$,

If moreover the following condition holds:

4. *the chain complex* $\mathrm{Ker}\,\kappa / \mathrm{Im}\,\iota$ *with boundary induced by the boundary of* D *is exact in degrees* $\geq m$.

The sequence is m-nearly exact*.*

Theorem

Given an m-nearly exact sequence of chain complexes

$$0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0,$$

there is a long exact sequence as follows:

$$\cdots \to \mathrm{H}_{m+1}(C) \xrightarrow{\iota} \mathrm{H}_{m+1}(D) \xrightarrow{\kappa} \mathrm{H}_{m+1}(E) \xrightarrow{\delta} \mathrm{H}_m(C) \to \mathrm{H}_m(D).$$

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Idea of proof: Filter the bicomplex by rows, then by columns and consider the corresponding spectral sequence.

Why is this useful for us? Because using another spectral sequences argument we obtain the desired relation between relative and usual Hochschild homologies.

Theorem

Let $A = B \oplus M$ be a split bounded extension. Let n be the index of B-tensor nilpotency of M and let $u = pdim_{B^e} M$. For any A-bimodule X, there is a Jacobi-Zariski long exact sequence as follows:

$$\cdots \to \mathrm{H}_{nu+1}(A, X) \xrightarrow{\kappa} \mathrm{H}_{nu+1}(A|B, X) \xrightarrow{\delta} \mathrm{H}_{nu}(B, X) \xrightarrow{\iota} \mathrm{H}_{nu}(A, X)$$

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So we get

Theorem

Let $A = B \oplus M$ be a split bounded extension of finite dimensional algebras. We have:

$$A \in \mathcal{H} \iff B \in \mathcal{H}.$$

More precisely we prove

1. $\mathrm{H}_*(A, A) = 0$ *for all* $* >> 0 \iff \mathrm{H}_*(B, B) = 0$ *for all* $* >> 0$.
2. A *is smooth if and only if* B *is smooth.*

Non split extensions

We would like to be able to use this procedure for other families of algebras and in order to do so we need to get rid of the **splitting** hypothesis.

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Caution! A/B has no multiplicative structure. So, even if each summand appearing in the differential $d : A \otimes_B (A/B)^{\otimes_B n+1} \otimes_B A \to A \otimes_B (A/B)^{\otimes_B n} \otimes_B A$ is not well defined, the complete expression of d is well-defined indeed. The differential is:

$$d(a_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{n-1} \otimes a_n) = a_0\sigma(\alpha_1) \otimes \cdots \otimes \alpha_{n-1} \otimes a_n +$$

$$\sum_{i=1}^{n-2} (-1)^i a_0 \otimes \cdots \otimes \pi(\sigma(\alpha_i)\sigma(\alpha_{i+1})) \otimes \cdots \otimes a_n + (-1)^{n-1} a_0 \otimes \alpha_1 \otimes \cdots \otimes \sigma(\alpha_{n-1}) a_n$$

where σ is a k-linear section of π.

Now, we have a short nearly exact sequence as in the split case and the corresponding spectral sequence converges to the homology of the complex $(\operatorname{Ker}\kappa/\operatorname{Im}\iota)_*$ (possibly non zero in infinitely many degrees).

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In particular we get a long nearly exact sequence (meaning that it is exact except maybe at $\mathrm{H}_*(A,X)$)

$$\cdots \to \mathrm{H}_i(B,X) \xrightarrow{\iota} \mathrm{H}_i(A,X) \xrightarrow{\kappa} \mathrm{H}_i(A|B,X) \xrightarrow{\delta} \mathrm{H}_{i-1}(B,X) \to \cdots \to \mathrm{H}_1(A|B,X)$$

From this, we are able to consider situations like those appearing in the following examples.

Example

$$
\begin{tikzcd}
 & & e_2 \arrow[dr, "a_2"] & \\
e_0 \arrow[r, "a_0"] & e_1 \arrow[ur, "a_1"] & & e_3 \\
 & & e_4 \arrow[ur, "a_3"'] &
\end{tikzcd}
\qquad
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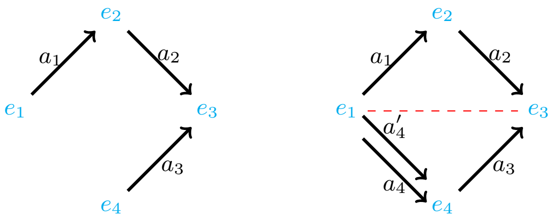
\end{tikzcd}
$$

B corresponds to the first quiver and A to the second one with the commutation relation $a_2a_1 - a_3a_4$. Here

$$
{}_B(A/B)_B = \langle a_4 \rangle = {}_B(S_3 \otimes P_1)_B,
$$

so it is right B-projective and it is B-tensor nilpotent.

Example



B corresponds to the first quiver and A to the second one with the commutation relations $a_2a_1 - a_3a_4 = 0$ and $a_2a_1 - a_3a_4' = 0$. Here

$$ {}_B(A/B)_B = \langle a_4, a_4' \rangle =_B (S_3 \otimes P_1)_B^2, $$

so it is right B-projective and it is B-tensor nilpotent.

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B corresponds to the first quiver and A to the second one with the commutation relation $a_2 a_1 a_0 - a_3 a_4 a_0 = 0$. Here

$$_B(A/B)_B = \langle a_4 \rangle.$$

It is B-tensor nilpotent but it is neither projective as right nor as left B-module since

$$_B(A/B) = S_3 \oplus P_3 \text{ and } (A/B)_B = S_1 \oplus P_1.$$

Thank you!