Derived equivalences for skew-gentle algebras.

Claire Amiot, joint with Thomas Brüstle

FD seminar, July 2020

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2. to Λ a gentle algebra, one can associate a marked surface with a collection of arcs [Opper-Plamondon-Schroll 2018], and algebraic properties of $\mathcal{D}^b(\Lambda)$ can be interpreted using the geometry of the marked surface [A-Plamondon-Schroll, Opper 2019].

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Question

Let Λ and Λ' be gentle algebras with a certain action of a group G. Can we find a geometric interpretation of the fact that ΛG and $\Lambda' G$ have the same derived category ?

Notation : k field, G finite abelian group such that $|G|$ invertible in k, Λ finite dimensional k-algebra with a G-action by automorphism.

We define ΛG as

$$\Lambda G = \Lambda \otimes kG \textit{ and } (\lambda \otimes g).(\lambda' \otimes g') = \lambda g(\lambda') \otimes gg'.$$

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Since ΛG is a natural left Λ-module, we get adjoint functors

$$\mathcal{D}^b(\Lambda) \underset{Res}{\overset{- \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda G}{\rightleftarrows}} \mathcal{D}^b(\Lambda G)$$

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Let $\widehat{G} = \mathrm{Hom}(G, k^*)$ be the dual group. Then $\widehat{G}$ acts on ΛG by

$$\chi.(\lambda \otimes g) = \chi(g)\lambda \otimes g$$

Proposition (RR'85)

The algebras $(\Lambda G)\widehat{G}$ and Λ are Morita equivalent.

Example

Let $\Lambda = k$ with trivial action of $G = \mathbb{Z}/2\mathbb{Z}$.
Then $\Lambda G = k \times k$. The action of $\widehat{G}$ exchanges the two copies of k.
$\Lambda G \widehat{G} = \mathrm{Mat}_2(k)$. It is Morita equivalent to k.

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Example

$$\Lambda = \begin{array}{ccccccc} & & 2^+ & \longrightarrow & 3^+ & \longrightarrow & 4^+ \\ & \nearrow & & & & & \\ 1 & & & & & & \\ & \searrow & & & & & \\ & & 2^- & \longrightarrow & 3^- & \longrightarrow & 4^- \end{array}$$

$$\Lambda G \sim \begin{array}{ccccccc} 1_0 & & & & & & \\ & \searrow & & & & & \\ & & 2 & \longrightarrow & 3 & \longrightarrow & 4 \\ & \nearrow & & & & & \\ 1_1 & & & & & & \end{array}$$

Definition

An object $T \in \mathcal{D}^b(\Lambda)$ is called *tilting* if

$$\forall i \neq 0,\ \mathrm{Ext}^i(T, T) = 0 \quad \text{and} \quad \mathrm{thick}(T) = \mathcal{D}^b(\Lambda).$$

Theorem (Happel-Rickard)

Let Λ and Λ' be finite dimensional algebras. Then $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\Lambda')$ if and only if there exists a tilting object $T \in \mathcal{D}^b(\Lambda)$ such that $\mathrm{End}(T) \simeq \Lambda$.

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Facts :

- If $T \in \mathcal{D}^b(\Lambda)$ is G-invariant, then $\mathrm{End}(T)$ has a natural G-action.
- If T is tilting G-invariant, then $T \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda G$ is tilting $\widehat{G}$-invariant.

Theorem (A-Brüstle)

1. *Let Λ be an algebra with G-actions, then we have*

$$\{ \text{ } G\text{-tilting subcat. of } \mathcal{D}^b(\Lambda)\} \stackrel{1-1}{\longleftrightarrow} \{ \text{ } \widehat{G}\text{-tilting subcat. of } \mathcal{D}^b(\Lambda G)\}$$

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$$\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda') \Rightarrow \mathcal{D}^b(\Lambda G) \underset{\widehat{G}}{\sim} \mathcal{D}^b(\Lambda' G).$$

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Remark

$$\mathcal{D}^b(\Lambda G) \underset{\widehat{G}}{\sim} \mathcal{D}^b(\Lambda' G) \Rightarrow \mathcal{D}^b(\Lambda G \widehat{G}) \underset{G}{\sim} \mathcal{D}^b(\Lambda' G \widehat{G}) \Rightarrow \mathcal{D}^b(\Lambda) \sim \mathcal{D}^b(\Lambda').$$

But it is not clear that it implies $\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda')$.

Let (S, M, P) be a marked surface. $M \subset \partial S$. A **dissection** D on (S, M, P) is a maximal collection of non intersecting arcs with endpoints in M or P, that do not cut out a subsurface of S.

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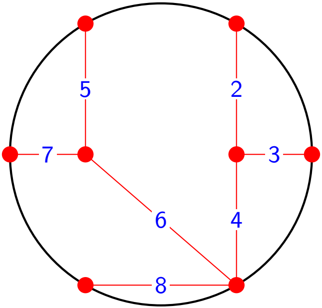
Theorem (APS-O '19)

Let Λ and Λ' be gentle algebras associated with (S, M, P, D) and (S', M', P', D') resp. The following are equivalent

1. $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\Lambda')$;
2. (S, M, P, η) *and* (S', M', P', η') *are diffeomorphic.*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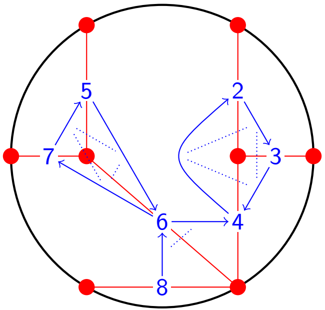
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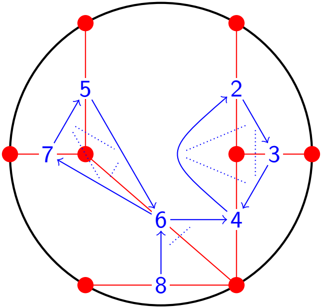
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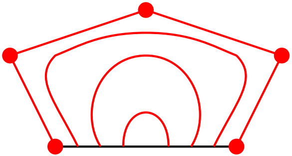

Let $\sigma \in \mathrm{Homeo}^+(S)$ of order 2 with finitely many fixed points such that $\sigma(M) = M$, $\sigma(P) = P$ and $\sigma(D) = D$. This defines a $\mathbb{Z}/2\mathbb{Z}$-action on Λ.

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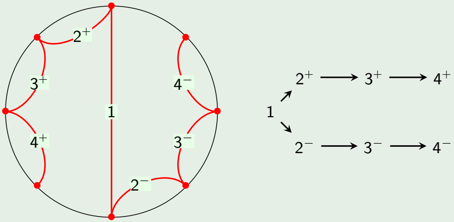
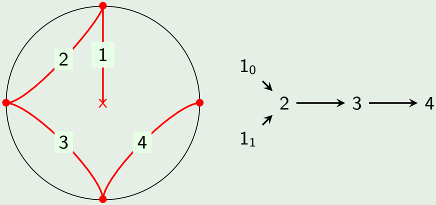
Example

(S, M)



$(\bar{S}, \bar{M}) = (S, M)/\sigma$



Proposition (AB)

ΛG is a skew-gentle algebra. All skew-gentle algebras are obtained in this way.

Skew-gentle algebras :[Geiss-de la Peña '95]. contains all gentle algebras, and D_n, $\widetilde{D}_n$ quivers.

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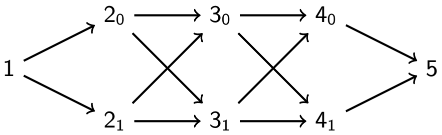
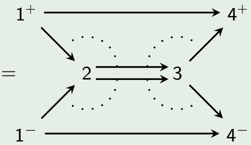


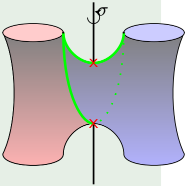
But, if A is skew-gentle, then (S, M, D) is not unique.

Example

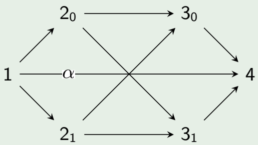
$\chi_1(\alpha) = \alpha$

$\Lambda_1 =$



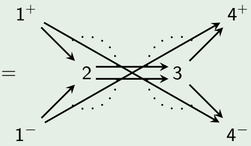


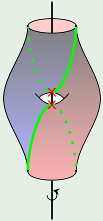
$A =$



$\chi_2(\alpha) = -\alpha$

$\Lambda_2 =$





Theorem (AB'19)

Let A and A' be two skew-gentle algebras, and Λ and Λ' the corresponding G-gentle algebras. Then the following are equivalent :

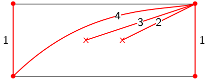
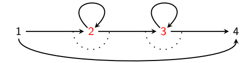
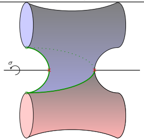
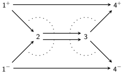
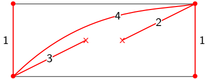
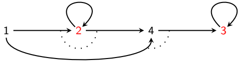
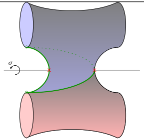
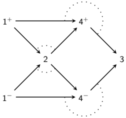
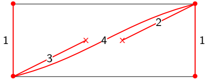
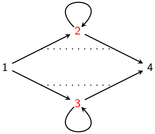
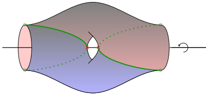
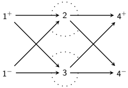
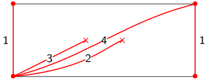
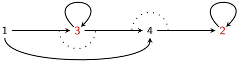
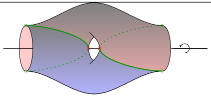
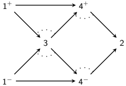
1. $\mathcal{D}^b(A) \underset{\widehat{G}}{\sim} \mathcal{D}^b(A')$.
2. $\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda')$
3. *there exists a G-diffeomorphism* $(S, M, P, \eta) \to (S', M', P', \eta')$.

Theorem (AB'19)

Let A and A' be two skew-gentle algebras. Then the following are equivalent :

1. *there exists a $\widehat{G}$-invariant tilting object T in $\mathcal{D}^b(A)$ with $\mathrm{End}(T) \simeq A'$;*
2. *there exists a homeomorphism $(\bar{S}, \bar{M}, \bar{\eta}) \to (\bar{S}', \bar{M}', \bar{\eta}')$.*

Here $\bar{S}$ is the orbifold S/σ.

$(\bar{S}, \bar{D})$	A	(S, σ)	Λ
			
			
			
			

Thank you very much

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