A geomteric model for the syzygies over certain 2-Calabi-Yau tilted algebras

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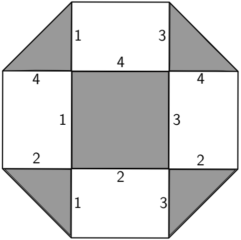
Overview — There is a correspondence

2-CY tilted algebra
of a certain type

$\longleftrightarrow$

Regular polygon with fixed
system of diagonals $\rho(x), x \in Q_0$

$$\begin{tikzcd}
1 \arrow[d] & 4 \arrow[l] \\
2 \arrow[r] & 3 \arrow[u]
\end{tikzcd}$$



projectifs $\begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\\4\end{smallmatrix}, \begin{smallmatrix}3\\4\\1\end{smallmatrix}, \begin{smallmatrix}4\\1\\2\end{smallmatrix}$

$$\begin{tikzcd}
 & P_3 \arrow[rr] \arrow[dr] & & P_2 \arrow[rr] \arrow[dr] & & P_4 \arrow[rr] \arrow[dr] & & P_3 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr] \arrow[dr] & & P_4 \arrow[rr] \arrow[dr] & & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[dr] & \\
1 \arrow[ur] & & \begin{smallmatrix}3\\4\end{smallmatrix} \arrow[ur] & & 2 \arrow[ur] & & \begin{smallmatrix}4\\1\end{smallmatrix} \arrow[ur] & & 3 \arrow[ur] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[ur] & & 4 \arrow[ur] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ur] & & 1
\end{tikzcd}$$

Periodic projective resolution

Overview — There is a correspondence

2-CY tilted algebra of a certain type $\longleftrightarrow$ Regular polygon with fixed system of diagonals $\rho(x), x \in Q_0$

non-projective syzygies $\longleftrightarrow$ 2-diagonals

AR translation $\longleftrightarrow$ Rotation R^2

Irred. morph. $\longleftrightarrow$ 2-pivots

Overview — Previous work

- Bastian-Holm-Ladkani (2013). Classification of derived equivalence classes of cluster-tilted algebras of Dynkin type.
- Chen-Geng-Lu (2015). Classification of the syzygy categories of cluster-tilted algebras of Dynkin type.
  - case by case analysis, using [BHL]
  - unions of $\underline{\mathrm{mod}}\,\Lambda_n$, where $\Lambda_n = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend right] \end{tikzcd} / \mathrm{rad}^{n-1}$
- Baur-Marsh (2008). Geometric model for 2-cluster categories $\mathcal{C}^2_{\mathbb{A}_n}$ via 2-diagonals in a $(2n+4)$-gon.
- Observation $\underline{\mathrm{mod}}\,\Lambda_n \cong \mathcal{C}^2_{\mathbb{A}_{n-2}}$

$\Rightarrow$ We should be able to think of syzygies as 2-diagonals in a regular polygon.

Our Motivation: Find an explicit and more general construction.

Plan

Definitions et recollections

The construction

Conjecture and Theorem

Syzygies

- M syzygy $\iff$ $M \subset P$ projective
- CMP B = category of syzygies over B
- $\underline{\text{CMP}}$ B = stable category

Exemple

$$\begin{aligned} B \text{ hereditary} \quad &\Rightarrow \text{CMP } B = \text{proj } B \\ &\Rightarrow \underline{\text{CMP}}\ B \text{ is trivial} \end{aligned}$$

because submodules of projectives are projective.

Syzygies

Let $N \in \operatorname{mod} B$ and $f\colon P(N) \to N$ a fixed projective cover. Then $\Omega\, N = \ker f$ is called the *syzygy of* N.

$$\begin{tikzcd} 0 \arrow[r] & \Omega\, N \arrow[r] & P(N) \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

$$M \text{ is a syzygy over } B \Leftrightarrow \exists N \; s.t. \; M = \Omega N$$

Example

B 2-Calabi-Yau tilted $\Rightarrow M \in \text{CMP } B \Leftrightarrow \text{Ext}^1_B(M, B) = 0$

- The syzygies over B are the (maximal) Cohen-Macauley modules over B.
- $\underline{CMP}(B)$ is a triangulated category with shift Ω.

Plan

Definitions et recollections

The construction

Conjecture and Theorem

From the quiver Q to the checkerboard polygon $\mathcal{S}$

The algebra B will be given by a quiver Q with potential. We construct a checkerboard polygon $\mathcal{S}$ in three steps.

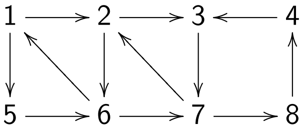
$$\begin{tikzcd}
Q \arrow[r, rightsquigarrow] & G \arrow[r, rightsquigarrow] & \widetilde{G} \arrow[r, rightsquigarrow] & \mathcal{S} \\
\text{quiver} & \text{dual graph} & \text{twisted dual graph} & \text{checkerboard polygon}
\end{tikzcd}$$

The quiver Q

Let Q be a quiver without loops and 2-cycles s.t.

- Q has no parallel arrows $\bullet \rightrightarrows \bullet$
- Q is planar
- faces of Q = oriented chordless cycles in Q
- for each arrow α
  - either α lies in a unique chordless cycle   boundary arrows
  - or α lies in exactly two chordless cycles   interior arrows
- Potential W = sum of all chordless cycles

Example.

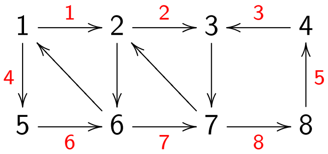


The dual graph G of Q

$G = (G_0, G_1)$

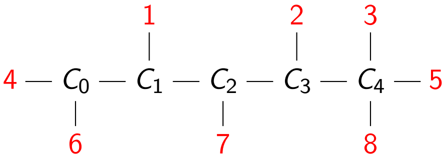
- $G_0 = \{\text{chordless cycles in } Q\} \cup \{\text{boundary arrows of } Q\}$
- G_1
  - C—C' if the two chordless cycles C, C' share an arrow
    trunk edges
  - C—α if α is a boundary arrow in the chordless cycle C
    leaf edges

Example.



The dual graph G of Q

Example.

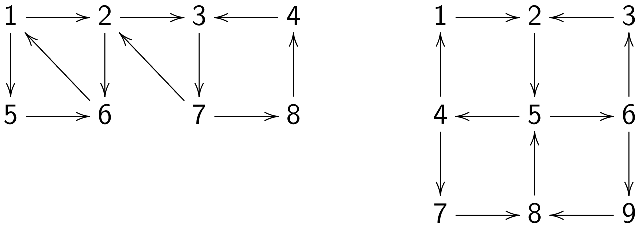


Remark: Additional condition on Q

Q is such that its dual graph G is a tree ($=$ connected, no cycles).

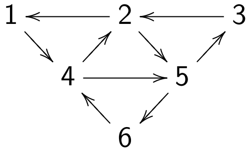
This means for each pair of chordless cycles C, C' of Q there exists a unique sequence of chordless cycles $C = C_1, C_2, \ldots, C_t = C'$ such that C_i and C_{i+1} share an arrow.

Thus we exclude, for example, the following quivers.



Remark: Additional condition on Q

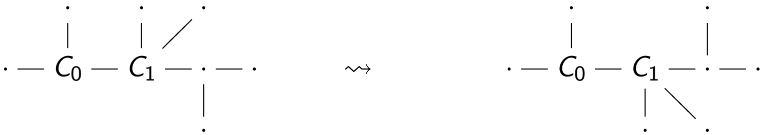
This one is also excluded.



The completed **twisted** dual graph $\widetilde{G}$

G is a tree. We choose a root C_0 such that C_0 is a chordless cycle that has at most one neighbor in the trunk. We are going to twist the graph G along every edge of the trunk starting at the edge C_0—C_1.

Example of the twist along C_0—C_1.

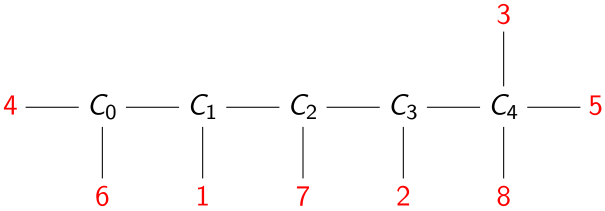


The **completed** twisted dual graph $\widetilde{G}$

Then we are connecting two neighboring leaves of the graph

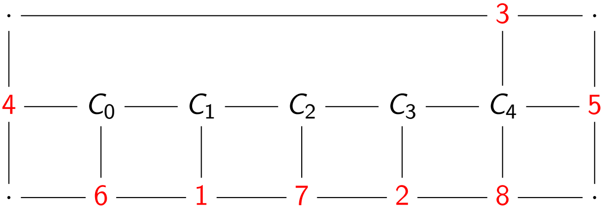
- by a new edge, if it produces a face with an even number of vertices;
- to a new vertex by adding two new edges, otherwise.

Example.

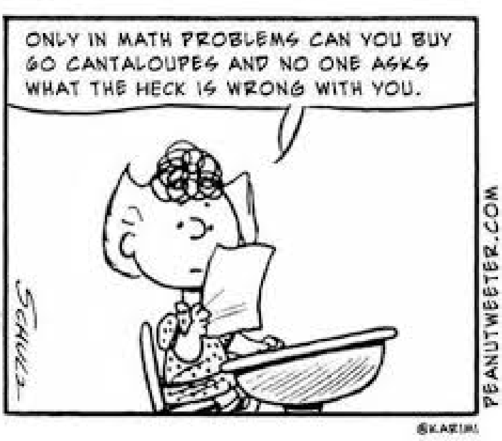


The **completed** twisted dual graph $\widetilde{G}$

Example.



...if this seems arbitrary to you so far, you are not alone...



One more step
!

The polygon $\mathcal{S}$

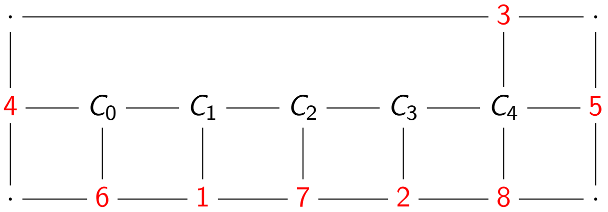
So far we have
Q quiver $\rightsquigarrow$ G dual graph $\rightsquigarrow$ $\widetilde{G}$ completed twisted graph

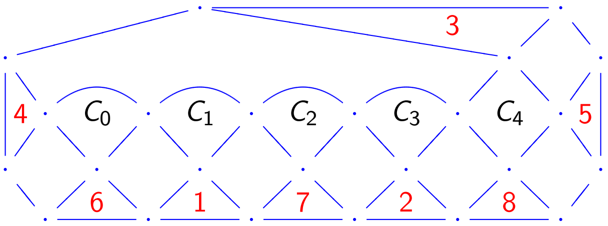
The last step is to construct the polygon $\mathcal{S}$ using the medial graph of $\widetilde{G}$.

The vertices of the **medial graph** are the edges of $\widetilde{G}$, and two vertices are connected if the corresponding edges are consecutive in a face of $\widetilde{G}$.

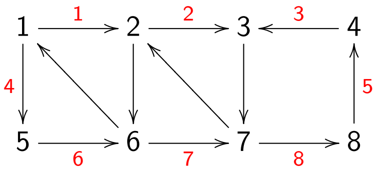
The polygon $\mathcal{S}$ is obtained from the medial graph of $\widetilde{G}$ by adding one edge for each leaf of G.

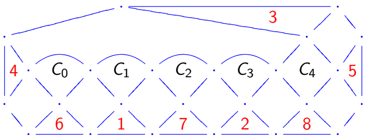
$\widetilde{G} \rightsquigarrow \mathcal{S}$

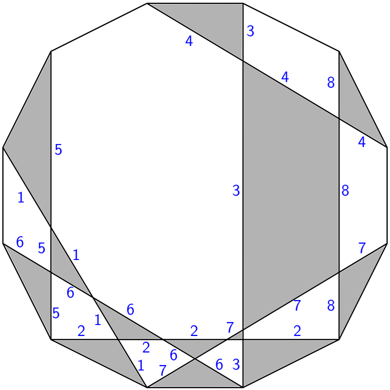




$Q \rightsquigarrow \mathcal{S}$







Properties of $\mathcal{S}$

- The intersection points in the checkerboard pattern of $\mathcal{S}$ are the arrows in Q.
- The shaded regions in the interior of $\mathcal{S}$ are the chordless cycles of Q.
- The shaded regions at the boundary of $\mathcal{S}$ are the boundary arrows of Q.
- The white regions have an even number of vertices and exactly one or two of them lie on the boundary of $\mathcal{S}$.
- The number of vertices of $\mathcal{S}$ is even. We label them clockwise $1, 2, 3, \ldots, 2N$.
- Each line $\rho(x), x \in Q_0$ of the checkerboard pattern is a 2-diagonal, i.e. it connects an even vertex to an odd vertex.

Orientation and degree

Let $\mathrm{Diag}(\mathcal{S}) = \{$oriented 2-diagonals of $\mathcal{S}\}$ where the orientation of the 2-diagonal is in the direction from the odd vertex to the even vertex.

Each $\gamma \in \mathrm{Diag}(\mathcal{S})$ crosses several checkerboard lines $\rho(x), x \in Q_0$. The **degree** of the crossing between γ and $\rho(x)$ is

$$\begin{cases} 0 & \text{if the crossing is from left to right;} \\ 1 & \text{if the crossing is from right to left.} \end{cases}$$

We define

$$P_0(\gamma) = \oplus P(x) \text{ sum over } x \text{ s.t. } \gamma \text{ crosses } \rho(x) \text{ in degree } 0;$$

$$P_1(\gamma) = \oplus P(x) \text{ sum over } x \text{ s.t. } \gamma \text{ crosses } \rho(x) \text{ in degree } 1.$$

2-diagonals ⇔ syzygies

Conjecture

For each 2-diagonal γ in $\mathcal{S}$ there exists a morphism

$$f_\gamma \colon P_1(\gamma) \to P_0(\gamma)$$

producing an equivalence of categories

$$\begin{aligned} F \colon \mathrm{Diag}(\mathcal{S}) \quad &\to \quad \underline{\mathrm{CMP}}\ B \\ \gamma \quad &\mapsto \quad \operatorname{coker} f_\gamma =: M_\gamma \quad \text{such that} \\ \rho(i) \quad &\leftrightarrow \quad \operatorname{rad} P(i) \\ R \quad &\leftrightarrow \quad \Omega \\ R^2 \quad &\leftrightarrow \quad \tau^{-1} = \text{Auslander-Reiten translation} \\ \text{2-pivots} \quad &\leftrightarrow \quad \text{irreducible morphisms} \end{aligned}$$

where R is the clockwise rotation by $2\pi/2N$.

2-pivots

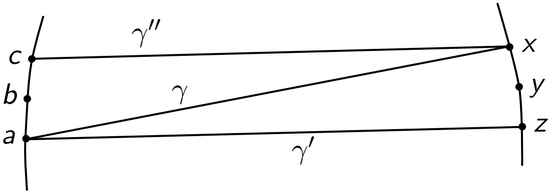


Figure: γ' is the 2-pivot of γ fixing the endpoint a and γ'' is the 2-pivot of γ fixing the endpoint b.

Corollary

Assuming the conjecture holds, the size $2N$ *of* $\mathcal{S}$ *is a derived invariant for the algebra* B *which can be computed combinatorially from the quiver* Q *of* B*.*

Main Result

Theorem (S.-Serhiyenko)

The conjecture holds if each chordless cycle is of length three.

Remark

1. The difficult part is to find the correct definition of $f_\gamma \colon P_1(\gamma) \to P_0(\gamma)$.
2. f_γ is not generic in general.
3. $M_\gamma = \operatorname{coker} f_\gamma$ is rigid, $\rightsquigarrow$ determined by its g-vector

Idea of the proof

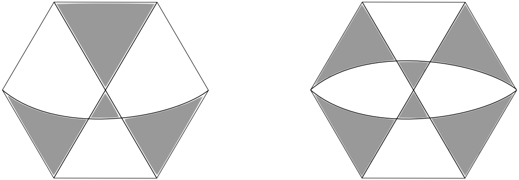
- Define f_γ.
- Show that $f_\gamma \circ f_{R(\gamma)}$ is exact. Thus $\Omega M_\gamma = M_{R(\gamma)}$
- Show that M_γ is indecomposable and independent of the choice of representative in the homotopy class of γ.
- Show that 2-pivots are irreducible morphisms.
- Show that there are no other syzygies.
  - 2-pivot meshes are Auslander-Reiten triangles. $\rightsquigarrow$ Diag($\mathcal{S}$) gives a finite component of the AR quiver of $\underline{\text{CMP}}$ B.
  - Show that there are no other components.

Corollary

Two of our algebras B, B' satisfy $\underline{\mathrm{CMP}}\ B \cong \underline{\mathrm{CMP}}\ B'$ if and only if the checkerboard polygons $\mathcal{S}, \mathcal{S}'$ have the same number of vertices.

Example.

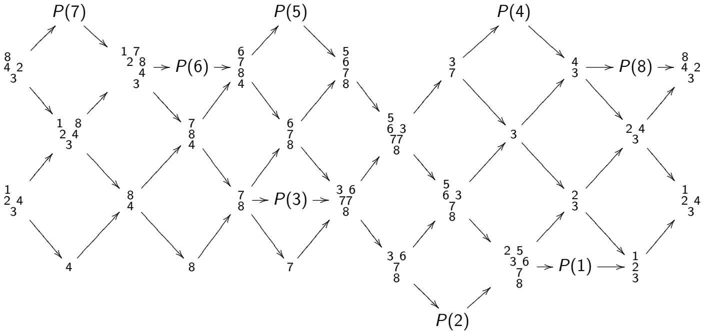
$$\mathrm{CMP}\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd}\right) \cong \mathrm{CMP}\left(\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \arrow[ll] \\ & 3 \arrow[ur] & \end{tikzcd}\right)$$



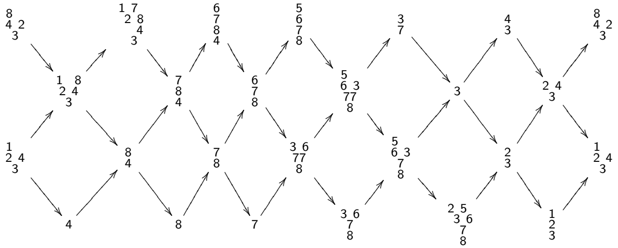
Current and future work

- General case, no restriction on the length of chordless cycles.
- Remove the condition
    - dual graph is a tree
        - not connected ✓
        - with cycles $\rightsquigarrow$ more complicated surfaces than polygons
    - faces of Q are chordless cycles
    - Q planar
    - Q without parallel arrows
- Study the effect of mutations on the checkerboard polygon
- Study tilting theory,
    - from $\mathrm{Diag}(\mathcal{S}) = \underline{\mathrm{CMP}}\ B$ to $\mathrm{mod}\ B$

CMP B



CMP *B*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