Simple objects in torsion-free classes over preprojective algebras of Dynkin type

Haruhisa Enomoto

18 June 2020, FD seminar (online)

Graduate School of Mathematics, Nagoya University

Overview

Today's talk

- Propose to study exact-categorical properties (simple objects, the Jordan-Hölder property) of torsion-free (or torsion) classes.
- Exhibit such study for preprojective algebra (and path algebra) using root system.

Outline

Simple objects and the Jordan-Hölder Property

Torsion-free classes over Preprojective algebras

Idea of Proof

Simple objects and the Jordan-Hölder Property

Setting

Throughout this talk,

- Λ: f.d. algebra over a field.
- $\operatorname{mod}\Lambda$: the cat. of f.g. right Λ-modules.

Definition

$\mathcal{E}$ is an exact category
if $\mathcal{E}$ is an extension-closed subcat. of $\operatorname{mod}\Lambda$, i.e.

$$0 \to L \to M \to N \to 0$$

s.e.s. in $\operatorname{mod}\Lambda$ with $L, N \in \mathcal{E}$ implies $M \in \mathcal{E}$.

Proj, inj, simples for exact cat.

For an exact category $\mathcal{E} \subset \operatorname{mod} \Lambda$,
a short exact sequence in $\mathcal{E}$ is a s.e.s. in $\operatorname{mod} \Lambda$

$$0 \to L \to M \to N \to 0$$

with $L, M, N \in \mathcal{E}$.

Definition

Let $\mathcal{E}$ be an exact category.

- $P \in \mathcal{E}$ is projective in $\mathcal{E}$ if every $P \to N$ lifts to $P \to M$.
- $I \in \mathcal{E}$ is injective in $\mathcal{E}$ if every $L \to I$ lifts to $M \to I$.
- $S \in \mathcal{E}$ is simple in $\mathcal{E}$ if for every s.e.s. $0 \to L \to S \to N \to 0$ in $\mathcal{E}$, we have $L = 0$ or $N = 0$.

Motivation

Let $\mathcal{F}$ be a funct-fin. torsion-free class in $\operatorname{mod}\Lambda$.
By Adachi-Iyama-Reiten, $\mathcal{F} = \operatorname{Sub} U$ for a s.τ^{-}-tilt. U.

- Projs in $\mathcal{F}$ are $\operatorname{add}(\Lambda/\operatorname{ann}\mathcal{F})$.
- Injs in $\mathcal{F}$ are $\operatorname{add} U$.

This implies $\#\{\text{indec. proj. in } \mathcal{F}\} = \#\{\text{indec. inj. in } \mathcal{F}\}$.

My Motivation is to study

$\operatorname{sim}\mathcal{F}$, the set of isoclasses of simples in $\mathcal{F}$,
for a given torsion-free class $\mathcal{F}$ in $\operatorname{mod}\Lambda$.

The Jordan-Hölder Property (JHP)

$\mathcal{E}$: an exact cat.

Definition

For an object $M \in \mathcal{E}$, a composition series of M in $\mathcal{E}$ is a sequence of submodules

$$0 = M_0 < M_1 < \cdots < M_m = M$$

satisfying $M_i / M_{i-1} \in \operatorname{sim} \mathcal{E}$ for each i.

Definition

$\mathcal{E}$ satisfies the Jordan-Hölder Property (JHP)
if for every $M \in \mathcal{E}$, all comp. ser. of M in $\mathcal{E}$ are equivalent, i.e.
$\mathcal{E}$-composition factors are unique up to perm.

Criterion for (JHP)

Theorem (E)

For a funct-fin. torsion-free class $\mathcal{F}$ in $\operatorname{mod}\Lambda$, TFAE:

1. $\mathcal{F}$ satisfies (JHP).
2. $\#\{\text{indec. proj. objects in } \mathcal{F}\} = \# \operatorname{sim} \mathcal{F}$.

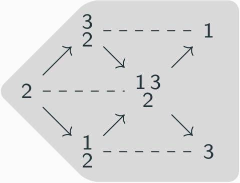
In general, $\#\{\text{indec. proj. objects in } \mathcal{F}\} \leq \# \operatorname{sim} \mathcal{F} \leq \infty$.

Example

Every torsion-free class over a Nakayama alg. satisfies (JHP).

Example

Let Λ be path alg. of $1 \to 2 \leftarrow 3$:

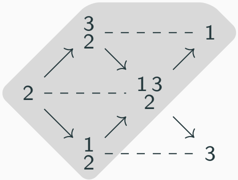


$\mathcal{F} := \mathsf{add}\{\text{gray}\}$.

Projectives in $\mathcal{F}$	$\begin{smallmatrix}1\\2\end{smallmatrix}$, 2, $\begin{smallmatrix}3\\2\end{smallmatrix}$
Simples in $\mathcal{F}$	1, 2, 3
(JHP)	

Example

Let Λ be path alg. of $1 \to 2 \leftarrow 3$:

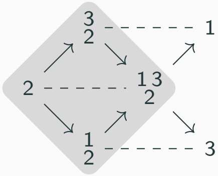


$\mathcal{F} := \mathsf{add}\{\text{gray}\}$.

Projectives in $\mathcal{F}$	$\begin{smallmatrix}1\\2\end{smallmatrix}$, 2, $\begin{smallmatrix}3\\2\end{smallmatrix}$
Simples in $\mathcal{F}$	1, 2, $\begin{smallmatrix}3\\2\end{smallmatrix}$
(JHP)	

Example

Let Λ be path alg. of $1 \to 2 \leftarrow 3$:



$\mathcal{F} := \mathsf{add}\{\text{gray}\}$.

Projectives in $\mathcal{F}$	$\begin{smallmatrix}1\\2\end{smallmatrix}$, 2, $\begin{smallmatrix}3\\2\end{smallmatrix}$
Simples in $\mathcal{F}$	2, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}3\\2\end{smallmatrix}$, $\begin{smallmatrix}1\,3\\2\end{smallmatrix}$
(JHP)	

Torsion-free classes over Preprojective algebras

Notation and the motivating theorem

From now on, we assume

- Q: a Dynkin quiver of type ADE.
- Φ: the root system of the same type as Q.
- Φ^+: the set of positive roots in Φ.
- α_u: the simple root corresponding to $u \in Q_0$.
- W: the Weyl group of Φ, generated by $s_u := s_{\alpha_u}$ for $u \in Q_0$.
- Π: a preprojective algebra of Q (defined later).
- $\operatorname{torf} \Lambda$: the poset of torsion-free classes (torfs) in $\operatorname{mod} \Lambda$.

Goal

Describe simples of torf over Π and kQ by using W and Φ!

Preprojective algebra

Definition

A preprojective algebra Π of Q is defined by

$$\Pi := k\overline{Q} \Big/ \left(\sum_{a \in Q_1} aa^* - a^* a \right) .$$

where $\overline{Q}$ is a double quiver and a^* is an added arrow.

Example

$Q : 1 \to 2 \leftarrow 3$

Some properties

Proposition

Π: *preproj. alg. of* Q.

1. Π *is f.d. self-injective alg (for Dynkin case).*
2. $\exists$ *natural surjection* $\Pi \twoheadrightarrow kQ$*, thus* $\operatorname{mod} kQ \subset \operatorname{mod} \Pi$.
3. Π *only depends on the underlying graph of* Q*, hence on* Φ*, and doesn't depend on the orientation.*

Bricks and simples

Let Λ be a f.d. alg and $\mathcal{F} \in \operatorname{torf} \Lambda$.

Proposition

Every simple object M in $\mathcal{F}$ is a brick,
i.e. every non-zero endomorphism of M is an isom.

Proof.

Let $f : M \to M$. Then we have s.e.s.

$$0 \to \ker f \to M \to \operatorname{Im} f \to 0$$

in $\mathcal{F}$ since $\mathcal{F}$ is closed under submodules.
Thus either $\ker f = 0$ ($\rightsquigarrow f$: isom) or $\operatorname{Im} f = 0$ ($\rightsquigarrow f = 0$). $\square$

Generalized Gabriel's theorem

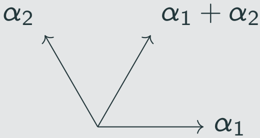
Define $\underline{\dim}\, M := \sum_{u\in Q_0}(\dim M_u)\alpha_u$ for $M \in \operatorname{mod}\Pi$.

Proposition (Iyama-Reading-Reiten-Thomas)

For every brick $B \in \operatorname{mod}\Pi$, we have $\underline{\dim}\, B \in \Phi^+$.

Example

$\overline{Q} : 1 \rightleftarrows 2.$



Torsion-free classes over preproj. alg.

Definition (Buan-Iyama-Reiten-Scott)

For $w \in W$, take a reduced expression $w = s_{u_1} s_{u_2} \cdots s_{u_l}$,
and define $\mathcal{F}(w) := \operatorname{Sub} \Pi / I(w) \subset \operatorname{mod} \Pi$, where

$$I(w) := I_{u_l} \cdots I_{u_2} I_{u_1},$$
$$I_u := \Pi(1 - e_u)\Pi.$$

Theorem (Mizuno)

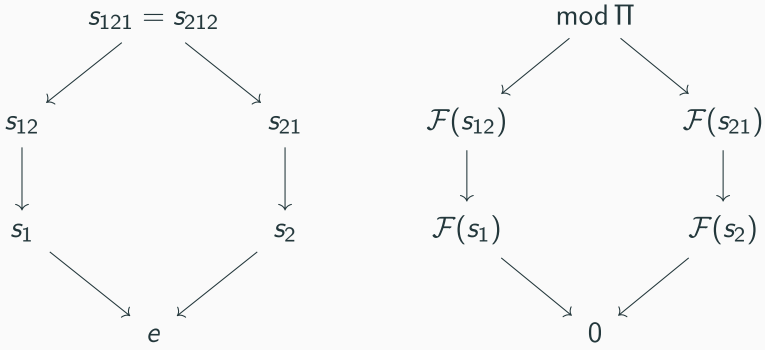
$w \mapsto \mathcal{F}(w)$ gives a bijection between W and $\operatorname{torf} \Pi$.

Remark

$\mathcal{F}(w) = \mathcal{C}_w$ categorifies the cluster structure of the unipotent cell in the algebraic group [Geiss-Leclerc-Schröer].

Example

$\overline{Q} : 1 \rightleftarrows 2.$



Hasse quivers of
right weak order $(W, \leq_R)$, and $\mathrm{torf}\,\Pi$.

Inversion set and torsion-free class

Definition

For $w \in W$, its inversion set is defined by

$$\mathrm{inv}(w) := \{\beta \in \Phi^+ \mid w^{-1}(\beta) \text{ is negative}\}.$$

$w_1 \leq_R w_2$ if and only if $\mathrm{inv}(w_1) \subseteq \mathrm{inv}(w_2)$

Proposition

For every brick $B \in \mathcal{F}(w)$, we have $\underline{\dim}\, B \in \mathrm{inv}(w)$.

$\rightsquigarrow$ *$\mathcal{F}(w)$ is a categorification of* $\mathrm{inv}(w)$

Example

$$\overline{Q} : 1 \rightleftarrows 2, \qquad \Phi^+ = \{\alpha_1, \alpha_2, \beta = \alpha_1 + \alpha_2\}.$$

$$\begin{array}{ccc} & s_{121} & \\ \swarrow & & \searrow \\ s_{12} & & s_{21} \\ \downarrow & & \downarrow \\ s_1 & & s_2 \\ \searrow & & \swarrow \\ & e & \end{array} \qquad \begin{array}{ccc} & \Phi^+ & \\ \rotatebox{45}{$\supset$} & & \rotatebox{135}{$\supset$} \\ \{\alpha_1, \beta\} & & \{\alpha_2, \beta\} \\ \cup & & \cup \\ \{\alpha_1\} & & \{\alpha_2\} \\ \rotatebox{135}{$\supset$} & & \rotatebox{45}{$\supset$} \\ & \varnothing & \end{array} \qquad \begin{array}{ccc} & \operatorname{mod} \Pi & \\ \swarrow & & \searrow \\ \mathcal{F}(s_{12}) & & \mathcal{F}(s_{21}) \\ \downarrow & & \downarrow \\ \mathcal{F}(s_1) & & \mathcal{F}(s_2) \\ \searrow & & \swarrow \\ & 0 & \end{array}$$

Bruhat inversions

Definition

For $w \in W$, its Bruhat inversion is $\beta \in \mathrm{inv}(w)$ which can't be written as a sum of other inversions of w.

$\mathrm{Binv}(w)$: the set of Bruhat inversions of w.

- For w_0: longest element, $\mathrm{inv}(w_0) = \Phi^+$ and $\mathrm{Binv}(w_0) = \{\text{simple roots}\}$.
- Bruhat inversions of w: "simple roots" inside $\mathrm{inv}(w)$.

Main Results

Theorem (E)

For a preprojective algebra Π and $w \in W$, we have a bijection

$$\begin{array}{ccc} \operatorname{brick} \mathcal{F}(w) & \xrightarrow{\underline{\dim}} & \operatorname{inv}(w) \\ \cup & & \cup \\ \operatorname{sim} \mathcal{F}(w) & \xrightarrow{\sim} & \operatorname{Binv}(w) \end{array}$$

Corollary

$\mathcal{F}(w)$ satisfies (JHP) if and only if

$$\# \operatorname{Binv}(w) = \# \operatorname{supp}(w).$$

Here $\operatorname{supp}(w) := \{u \in Q_0 \mid s_u$ *appears in red. exp. of* $w\}$.

Path algebra case

For $w \in W$, define

$$\mathcal{F}_Q(w) := \mathcal{F}(w) \cap \operatorname{mod} kQ \subset \operatorname{mod} kQ.$$

Then $w \mapsto \mathcal{F}_Q(w)$ induces a bij. between c_Q-sortable elements in W and torfs in $\operatorname{mod} kQ$ by Ingalls-Thomas.

Then the same result holds: we have a bij
$\underline{\dim}\colon \operatorname{sim} \mathcal{F}_Q(w) \xrightarrow{\sim} \operatorname{Binv}(w)$.

Compute inversions

Proposition

Fix a red. exp. $w = s_{u_1} \cdots s_{u_l} \in W$*, put*

$$\beta_i = s_{u_1} \cdots s_{u_{i-1}}(\alpha_{u_i})$$

for $i = 1, 2, \ldots, l$*. Then* $\mathrm{inv}(w) = \{\beta_1, \ldots, \beta_l\}$.

Example

$w = s_{21323} = s_2 s_1 s_3 s_2 s_3$ for $\overline{Q} : 1 \rightleftarrows 2 \rightleftarrows 3$. Then

$$\beta_1 = \alpha_2, \qquad \beta_2 = s_2(\alpha_1) = \alpha_1 + \alpha_2,$$
$$\beta_3 = s_{21}(\alpha_3) = \alpha_2 + \alpha_3, \quad \beta_4 = s_{213}(\alpha_2) = \alpha_1 + \alpha_2 + \alpha_3,$$
$$\beta_5 = s_{2132}(\alpha_3) = \alpha_1.$$

Compute Bruhat inversions

Proposition

Fix a red. exp. $w = s_{u_1} \cdots s_{u_l} \in W$ *and* β_i *as before. Then TFAE:*

1. $\beta_i \in \mathrm{Binv}(w)$.
2. $s_{u_1} \cdots \widehat{s_{u_i}} \cdots s_{u_l}$ *(*s_{u_i} *omitted) is reduced.*

Example

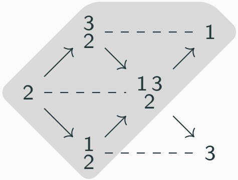
$w = s_{21323}$ for $\overline{Q} : 1 \rightleftarrows 2 \rightleftarrows 3$. Then

$$\beta_1 = \alpha_2, \quad \beta_2 = \alpha_1 + \alpha_2, \quad \beta_3 = \alpha_2 + \alpha_3,$$
$$\beta_4 = \alpha_1 + \alpha_2 + \alpha_3, \quad \beta_5 = \alpha_1.$$

$\hat{2}1323$:red, $2\hat{1}323$:not, $21\hat{3}23$:red, $213\hat{2}3$:not, $2132\hat{3}$:red.

Example

Let Λ be path alg. of $1 \to 2 \leftarrow 3$, $w = s_{21323}$.



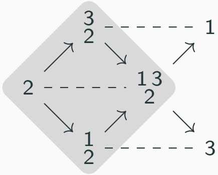
$\mathcal{F}_Q(w) = \mathsf{add}\{\text{gray}\}$.

$\hat{2}1323$:red, $2\hat{1}323$:not, $21\hat{3}23$:red, $213\hat{2}3$:not, $2132\hat{3}$:red.

2 $\begin{smallmatrix}1\\2\end{smallmatrix}$ $\begin{smallmatrix}3\\2\end{smallmatrix}$ $\begin{smallmatrix}1\,3\\2\end{smallmatrix}$ 1

Example for path alg. case

Let Λ be path alg. of $1 \to 2 \leftarrow 3$, $w = s_{2132}$.



$\mathcal{F}_Q(w) = \mathsf{add}\{\text{gray}\}$.

$\hat{2}132$:red, $2\hat{1}32$:red, $21\hat{3}2$:red, $213\hat{2}$:red.

2 $\quad$ $\begin{smallmatrix}1\\2\end{smallmatrix}$ $\quad$ $\begin{smallmatrix}3\\2\end{smallmatrix}$ $\quad$ $\begin{smallmatrix}1\,3\\2\end{smallmatrix}$

Idea of Proof

Brick sequence associated to red. exp.

From now on, fix one red. exp $w = s_{u_1} s_{u_2} \cdots s_{u_l}$.

By this data, we have the following chain in $\operatorname{torf} \Pi$.

$$0 = \mathcal{F}(e) \leftarrow \mathcal{F}(s_{u_1}) \leftarrow \mathcal{F}(s_{u_1} s_{u_2}) \leftarrow \cdots \leftarrow \mathcal{F}(s_{u_1} \cdots s_{u_l}) = \mathcal{F}(w)$$

Proposition (Demonet-Iyama-Reading-Reiten-Thomas)

For an arrow $\mathcal{G} \leftarrow \mathcal{F}$ in $\operatorname{torf} \Lambda$, there's a brick B such that $\mathcal{F} = \operatorname{Filt}\{\mathcal{G} \cup \{B\}\}$ (called brick label of this arrow).

Define $B_1, B_2, \ldots, B_l$ as brick labels of above arrows.

Corollary

$\mathcal{F}(w) = \operatorname{Filt}\{B_1, B_2, \ldots, B_l\}$, *hence* $\operatorname{sim} \mathcal{F}(w) \subset \{B_1, \ldots, B_l\}$.

Brick sequence and inversion sets

Proposition (Amiot-Iyama-Reiten-Todorov, layer module)

$\underline{\dim}\, B_i = \beta_i$, where $\beta_i = s_{u_1} \cdots s_{u_{i-1}}(\alpha_{u_i})$ as before. Thus

$$\operatorname{inv}(w) = \{\underline{\dim}\, B_1, \ldots, \underline{\dim}\, B_l\}.$$

$$\begin{array}{ccc} \operatorname{sim} \mathcal{F}(w) & \subset & \{B_1, \ldots, B_l\} \\ \downarrow & & \cong \big\downarrow \underline{\dim} \\ \operatorname{Binv}(w) & \subset & \operatorname{inv}(w) = \{\beta_1, \ldots, \beta_l\} \end{array}$$

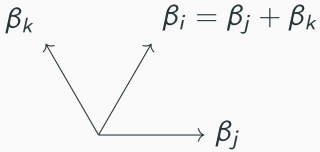
Hence suffices to show TFAE:

1. B_i is non-simple in $\mathcal{F}(w)$.
2. $\beta_i = \underline{\dim}\, B_i$ is a non-Bruhat inversion of w.

1. B_i is non-simple in $\mathcal{F}(w)$.
2. $\beta_i = \underline{\dim}\, B_i$ is a non-Bruhat inversion of w.

(1) $\Rightarrow$ (2): Easy by $\underline{\dim}$.

(2) $\Rightarrow$ (1): We use some geometrical configuration of non-Bruhat inversions:

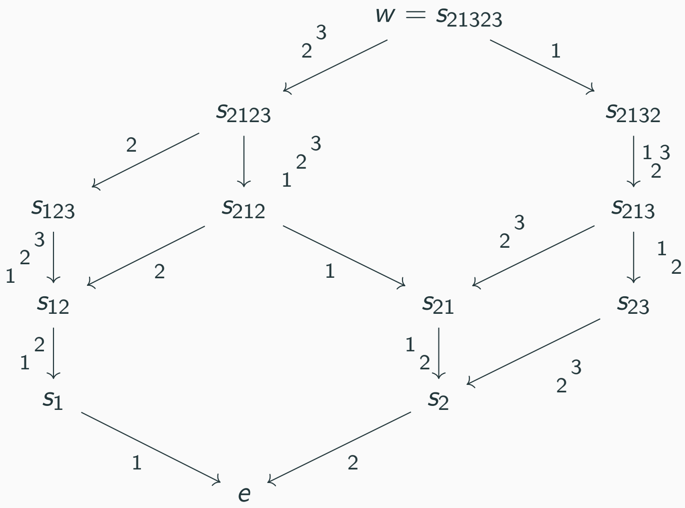


and a non-zero non-injection $f\colon B_i \to B_j$, which gives

$$0 \to \ker f \to B_i \to \operatorname{Im} f \to 0,$$

hence B_i is non-simple.

Brick sequences for several red. exp.



Conjectures

Conjecture

If $\underline{\dim}\, B_i$ is non-Bruhat inv. of w, then there's s.e.s.

$$0 \to B_j \to B_i \to B_k \to 0$$

with for some j, k.

This is (almost) equivalent to:

Conjecture

TFAE for a brick $B \in \mathcal{F}(w)$.

- B is simple in $\mathcal{F}(w)$.
- B appears as a label in every path from $\mathcal{F}(w)$ to 0.