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THE QUIVER OF *n*-HEREDITARY ALGEBRAS

FD Seminar

Louis-Philippe Thibault
based on joint work with Mads Hustad Sandøy

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Higher Auslander–Reiten theory

Slogan: Auslander–Reiten theory can be viewed as a 2-dimensional theory.

Example: Auslander Correspondence

There exists a bijection

$$\{\text{f.d. representation-finite algebras } \Lambda\} / \sim \;\overset{1:1}{\longleftrightarrow}\; \left\{\begin{matrix}\text{f.d. algebras } \Gamma \text{ satisfying} \\ \mathrm{gl.dim}\, \Gamma \leq 2 \leq \mathrm{dom.dim}\, \Gamma\end{matrix}\right\} / \sim$$

$$\Lambda \longmapsto \Gamma := \mathrm{End}_\Lambda(M)$$

where M is an additive generator of Λ.

n-Auslander Correspondence (Iyama 2007)

There exists a bijection

$$\left\{\begin{matrix}(n-1)\text{-cluster-tilting subcat.} \\ \mathcal{C} \subset \operatorname{mod}\Lambda \text{ for some f.d. algebra } \Lambda\end{matrix}\right\} / \sim \;\overset{1:1}{\longleftrightarrow}\; \left\{\begin{matrix}\text{f.d. algebras } \Gamma \text{ satisfying} \\ \mathrm{gl.dim}\,\Gamma \leq n \leq \mathrm{dom.dim}\,\Gamma\end{matrix}\right\} / \sim$$

$$\mathcal{C} \longmapsto \Gamma := \mathrm{End}_\Lambda(M)$$

where M is an additive generator of $\mathcal{C}$.

n-hereditary algebras

- Many key features (e.g. AR-translate and AR-sequence) of Auslander–Reiten theory have natural generalisations in a higher dimensional setting.
- n-hereditary algebras arise from this paradigm. They enjoy properties analogous to hereditary algebras in the classical theory ($n = 1$).

Let Λ be a f.d. algebra of finite global dimension n and $D := \operatorname{Hom}_k(-, k)$.

Nakayama functor

$$\nu := D\mathbf{R}\operatorname{Hom}_\Lambda(-, \Lambda) : \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$$

$$\nu^{-1} := \mathbf{R}\operatorname{Hom}_\Lambda(D\Lambda, -) : \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$$

This is a Serre functor on $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$, that is,

$$\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)}(X, Y) \cong D\operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)}(Y, \nu(X))$$

for any $X, Y \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$.

Auslander–Reiten translation

Denote $\nu_i := \nu \circ [-i]$. In the classical case $n = 1$, the *AR-translation* $\tau_1 := D\,\mathrm{Tr}$ is isomorphic to

$$\tau_1 := \mathrm{H}^0(\nu_1) = D\,\mathrm{Ext}^1_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda.$$

There is thus a natural higher dimensional generalisation:

$$\tau_n := \mathrm{H}^0(\nu_n) = D\,\mathrm{Ext}^n_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda.$$

Properties to generalise

- A key reason for which the usual definition of $\tau_1 = D\,\mathrm{Tr}$ agrees with $\mathrm{H}^0(\nu_1)$ in the case $n = 1$, thus giving an endofunctor of $\mathrm{mod}\,\Lambda$, is that
  - $\mathrm{Hom}_\Lambda(M, \Lambda) = 0 \quad \forall M$ non-projective;
  - $\mathrm{Hom}_\Lambda(D\Lambda, N) = 0 \quad \forall N$ non-injective.

In other words,

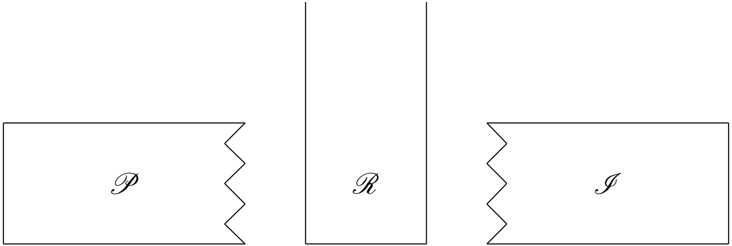
$$\nu_1^{-1}(N) \text{ is only concentrated in degree } 0 \quad \forall N \text{ non-injective.}$$

Properties to generalise

- One can distinguish between representation-finite and representation-infinite algebras as follows. Define

$$\mathscr{P} := \mathrm{add}\{\tau_1^{-i}(\Lambda) \mid i \geq 0\} \quad \text{and} \quad \mathscr{I} := \mathrm{add}\{\tau_1^{i}(D\Lambda) \mid i \geq 0\}$$

  the subcategories of preprojective and preinjective Λ-modules respectively. Then Λ is
  - *representation-finite* if and only if $\mathscr{P} = \mathscr{I}$;
  - *representation-infinite* if and only if $\mathscr{P} = \mathrm{add}\{\nu_1^{-i}(\Lambda) \mid i \geq 0\}$.



Definition

Let Λ be a finite-dimensional algebra of global dimension n. We say that Λ is

1. *n-representation-finite* if for all $P \in \operatorname{ind.proj}\Lambda$, there exists $i \geq 0$ such that $\nu_n^{-i}(P) \in \operatorname{ind.inj}\Lambda$;
2. *n-representation-infinite* if $\nu_n^{-i}(\Lambda)$ is concentrated in degree 0 for all $i \geq 0$;
3. *n-hereditary* if it is n-representation-finite or n-representation-infinite.

- In case **(1)**, $\Pi := \bigoplus_{i \geq 0} \tau_n^{-i}(\Lambda)$ is an n-cluster-tilting Λ-module, that is,

$$\begin{aligned} \operatorname{add}\Pi &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \Pi) = 0 \text{ for all } 0 < i < n\} \\ &= \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\Pi, Y) = 0 \text{ for all } 0 < i < n\} \end{aligned}$$

  and $\operatorname{add}\Pi = \mathscr{P} = \mathscr{I}$ ([Iyama '11]).
- In both cases, $\mathscr{P} \vee \mathscr{I}$ has n-almost split sequences ([Iyama '07, HIO '14]).

Classes of examples

n-representation-finite algebras

- [HI '11] Tensor products of ℓ-homogeneous higher representation-finite algebras are higher representation-finite.
- [IO '11] Higher type A algebras are n-representation-finite.
- [IO '13] Quasi-tilted algebras of canonical type $(2, 2, 2, 2)$ are 2-representation-finite.

Classes of examples

n-representation-infinite algebras

- [HIO '14] Tensor products of higher representation-infinite algebras are higher representation-infinite.
- [AIR '15] If $G < SL(n+1, k)$ is a finite cyclic group satisfying a certain condition, then there exists a grading on the skew-group algebra $k[x_0, \ldots, x_n]\#G$ such that the degree 0 part is n-representation-infinite. (Higher McKay correspondence)
- [HIO '14] Higher type $\tilde{A}$ algebras are n-representation-infinite.
- [BS '10] Let Z be a smooth projective Fano variety with $\dim Z = n$ and $T \in \mathrm{D^b}(\mathrm{Coh}\, Z)$ be a tilting object.
  Then $\Lambda = \mathrm{End}_Z(T)$ is n-representation-infinite.

Motivating Problem

- In the case $n = 1$, there is a complete classification of the representation-finite and representation-infinite finite-dimensional hereditary algebras (Gabriel).

Problem

Classify the n-hereditary algebras.

Questions

- Is the quiver of an n-hereditary algebra acyclic? (Conjecture: yes [HIO '14])
- Is there a bound on $\dim_k \operatorname{Ext}^1(S_i, S_j)$?
- Can we classify certain subclasses of n-hereditary algebras?

Some known classification results

- Iyama and Oppermann ('13) classified the iterated tilted 2-representation-finite algebras, using the classification of selfinjective cluster tilted algebras [Ringel '08].

- Vaso ('17) classified the n-representation-finite Nakayama algebras.

Formality

- Hereditary algebras are *formal*, that is, for any object $X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$,
$$X \cong \bigoplus_{\ell \in \mathbb{Z}} \mathsf{H}^{\ell}(X)[-\ell].$$
- There is an analogous property for n-hereditary algebras. Define
$$\mathsf{D}^{n\mathbb{Z}}(\operatorname{mod}\Lambda) := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod}\Lambda) \mid \mathsf{H}^{i}(X) = 0 \ \forall i \in \mathbb{Z} \setminus n\mathbb{Z}\}.$$
Suppose $\operatorname{gl.dim}\Lambda = n$. Then Λ is n-hereditary if and only if
$$\nu_n^i(\Lambda) \in \mathsf{D}^{n\mathbb{Z}}(\operatorname{mod}\Lambda) \quad \text{for all } i \in \mathbb{Z} \quad \text{[HIO '14]}.$$
In particular, this implies that
$$\nu_n^i(\Lambda) \cong \bigoplus_{\ell \in \mathbb{Z}} \mathsf{H}^{\ell n}(\nu_n^i(\Lambda))[-\ell n] \quad \text{for all } i \in \mathbb{Z} \quad \text{[Iyama '11]}$$
and
$$\operatorname{Ext}_{\Lambda}^{\ell}(D\Lambda, \Lambda) = 0 \quad \text{for all } 0 < \ell < n.$$

Formality as an obstruction

- Formality is a very good first obstruction, allowing us to narrow the subclass of n-hereditary algebras by quite a lot.

Lemmata

Let $\Lambda = kQ/I$ be a finite-dimensional algebra.
Suppose that $\mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) = 0$. Then

- Every arrow in Q is part of a relation.

If, in addition, Λ is monomial, then

- Every relation r which does not start at a source and end at a sink must intersect with at least one other relation;
- For every sink (resp. source) vertex i, there is exactly one arrow a such that $h(a) = i$ (resp. $t(a) = i$).

Truncated path algebras

- We obtain another consequence of formality for truncated path algebras.

Theorem

Let Q be a finite quiver and $J \subset kQ$ the arrow ideal. Let $\Lambda = kQ/J^{\ell}$ for some $\ell \geq 2$. Suppose that $\mathrm{Ext}^1_{\Lambda}(D\Lambda, \Lambda) = 0$. Then Λ is a Nakayama algebra.

- Using Vaso's classification of the n-representation-finite Nakayama algebras, we obtain the following corollary.

Let $\mathbb{A}_m$ be the linearly oriented Dynkin quiver of type A with m vertices:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow m-1 \longrightarrow m$$

Corollary

Let Q be a finite quiver and $J \subset kQ$ the arrow ideal. Let $\Lambda = kQ/J^\ell$ for some $\ell \geq 2$. The following are equivalent:

1. Λ is n-hereditary;
2. $Q = \mathbb{A}_m$ and $\ell \mid (m-1)$ or $\ell = 2$.

In this case, $n = 2\frac{m-1}{\ell}$ and Λ is an n-representation-finite algebra.

Preprojective algebras

- A useful perspective in understanding n-hereditary algebras is to consider their preprojective algebra

$$\Pi := \bigoplus_{i \geq 0} \tau_n^{-i}(\Lambda).$$

- If Λ is n-representation-finite, then Π is a selfinjective algebra. The converse is true if $n = 2$. Moreover, $\underline{\mathrm{mod}}\,\Pi$ is an $(n+1)$-Calabi–Yau category ([IO '13]).
- Λ is n-representation-infinite if and only if Π is a bimodule Calabi–Yau algebra of Gorenstein parameter 1. In this case, $\mathrm{D}^{\mathrm{fd}}(\mathrm{mod}\,\Pi)$ is an $(n+1)$-Calabi–Yau category ([AIR '15]).

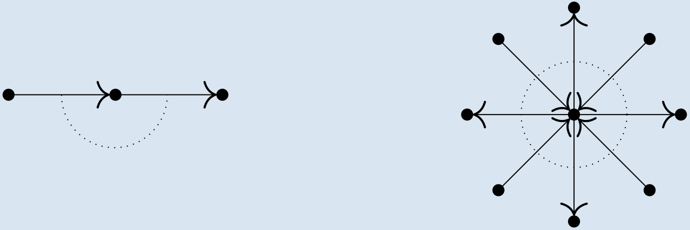
Quadratic monomial 2-hereditary algebras

- We restrict to the case of quadratic monomial 2-hereditary algebras.
- The preprojective algebras over 2-hereditary algebras enjoy an extra useful property: they are Jacobian algebras whose relations are encoded in a potential ([Keller ’11]).

Quadratic monomial 2-hereditary algebras

Theorem

Let $\Lambda = kQ/I$ be a 2-hereditary quadratic monomial algebra. Then Λ is one of the following two bound quiver algebras:



These algebras are 2-representation-finite.

Remark

The second algebra can be obtained by taking a 2-APR-tilt of $A_3 \otimes_k A_3$.

What we can deduce from formality

Proposition

Let $\Lambda = kQ/I$ be a quadratic monomial algebra of global dimension 2. Suppose that $\mathrm{Ext}^1_\Lambda(D\Lambda, \Lambda) = 0$. Then Q is a quiver of the form: