Calabi–Yau properties of Postnikov diagrams

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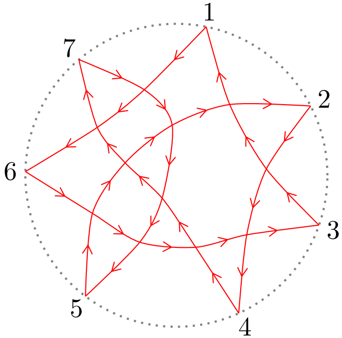
Postnikov diagrams

A Postnikov diagram D consists of n oriented strands in an oriented disc, connecting marked points $\{1, \ldots, n\}$ around the boundary, and satisfying



(P0) Each marked point is the source of one strand and the target of one strand.

(P1) The strands cross transversely, pairwise, and finitely many times.

(P2) Moving along each strand, the signs of its crossings with other strands alternate.

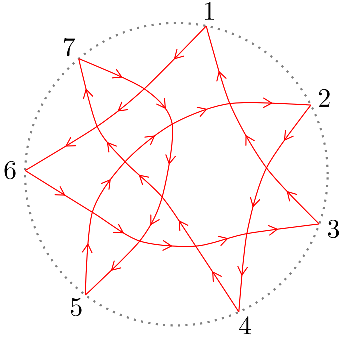
(P3) A strand does not cross itself.

(P4) If two strands cross twice, they are oriented in opposite directions between these crossings.

D determines $\sigma_D \in \mathfrak{S}_n$ by mapping the source of each strand to its target. In the example, $\sigma_D = (1, 6, 3)(2, 4, 7, 5)$.

The quiver

The strands of D cut the disc into regions, such that the orientation of strands around the boundary of each region is either *alternating*, *clockwise*, or *anticlockwise*.



D determines a quiver Q_D with

(Q_0) vertices corresponding to the alternating regions

(Q_1) arrows corresponding to crossings of strands

Some vertices and arrows are on the boundary, and will sometimes play a different role to the others—we mark them in blue and call them *frozen*.

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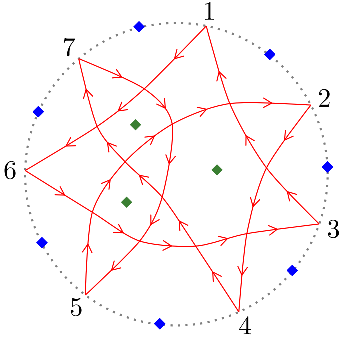

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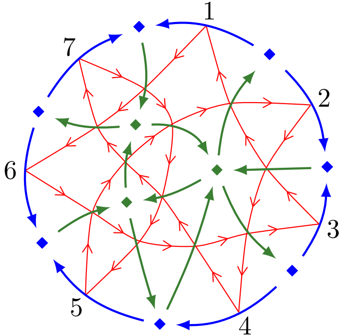

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Two commutative algebras

The permutation σ_D is a Grassmann permutation, and hence determines a particular *positroid* subvariety $\Pi^\circ(\sigma_D) \subseteq \mathrm{Gr}_k^n$ of the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ [Postnikov].

Our first commutative algebra is the homogeneous coordinate ring $\mathbb{C}[\widehat{\Pi}^\circ(\sigma_D)]$ of this projective variety.

Our second is the cluster algebra $\mathscr{A}_D$ with invertible frozen variables determined by the quiver Q_D.

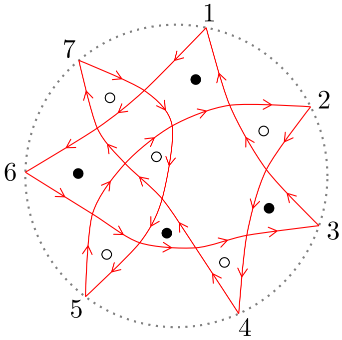
Theorem (Serhiyenko–Sherman-Bennett–Williams, Galashin–Lam)

There is an isomorphism $\mathscr{A}_D \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ(\sigma_D)]$*, mapping the initial cluster variables to restrictions of Plücker coordinates.*

In particular, $\mathscr{A}_D$ depends only on σ_D; the choice of D corresponds to a choice of initial seed.

For $\sigma_D \colon i \mapsto i + k \mod n$, the variety $\Pi^\circ(\sigma_D)$ is dense in Gr_k^n, and the cluster algebra with non-invertible frozen variables attached to Q_D is isomorphic to the homogeneous coordinate ring $\mathbb{C}[\widehat{\mathrm{Gr}_k^n}]$. [Scott]

A non-commutative algebra



The oriented regions of D are either clockwise ($\circ$) or anticlockwise ($\bullet$).

Thus Q_D has a determined set of $\bullet$-cycles and $\circ$-cycles.

Let A_D be the $\mathbb{C}$-algebra determined by Q_D with relations as follows:

Each non-boundary (green) arrow a can be completed to either a $\bullet$-cycle or a $\circ$-cycle by unique paths $p_a^\bullet$ and p_a°; we impose the relation $p_a^\bullet = p_a^\circ$ for each a.

This is an example of a *frozen Jacobian algebra*, for the potential $W = \sum(\bullet\text{-cycles}) - \sum(\circ\text{-cycles})$.

Technical note: we take the complete path algebra of Q_D over $\mathbb{C}$, and the quotient by the closure of the ideal generated by the given relations.

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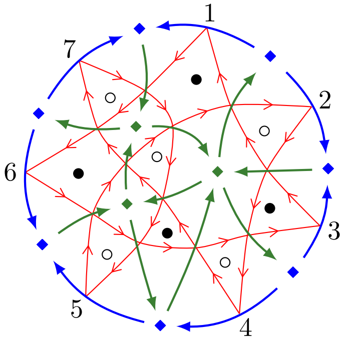

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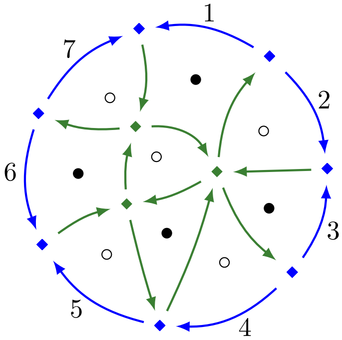

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Main goal

Objective

Use the non-commutative algebra A_D to construct an additive categorification of the cluster algebra $\mathscr{A}_D \xrightarrow{\sim} \mathbb{C}[\widehat{\Pi}^\circ(\sigma_D)]$.

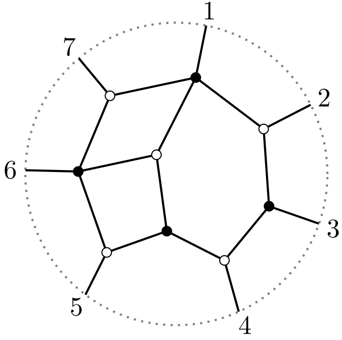
Our approach will hinge on a particular Calabi–Yau symmetry property of the algebra A_D, which we will come to shortly.

This requires a non-degeneracy assumption: we ask that D is *connected*, meaning the union of its strands is a connected set.

For the special permutation $\sigma_D \colon i \mapsto i + k \mod n$, a categorification of $\mathscr{A}_D$ is provided by Jensen–King–Su's Grassmannian cluster category. We will recover this category for these special diagrams, but via a different approach.

Interlude: dimer models

Consider a bipartite graph drawn in our disc, together with *half-edges* connecting some nodes to the boundary marked points.



This is called a dimer model, and it also determines a quiver and frozen Jacobian algebra, called the *dimer algebra*.

This construction makes sense on any oriented surface with or without boundary.

Theorem (Broomhead)

The dimer algebra of a consistent dimer model on the torus is bimodule 3-Calabi–Yau.

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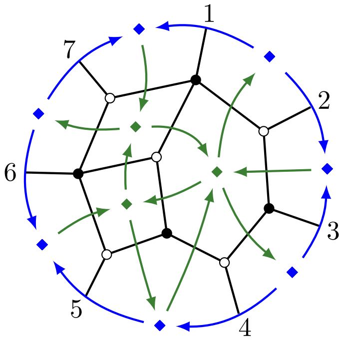

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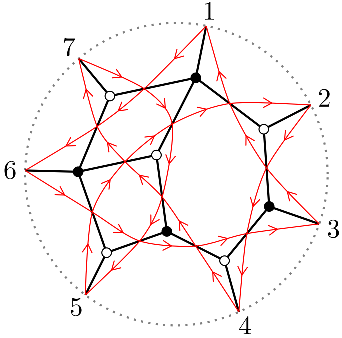

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Internally Calabi–Yau algebras

Our main result is a version of Broomhead's theorem, adapted to dimer models on the disc by weakening the 3-Calabi–Yau property at the boundary.

Let A be a Noetherian $\mathbb{K}$-algebra, $e = e^2 \in A$ an idempotent, and $A^{\varepsilon} = A \otimes_{\mathbb{C}} A^{\mathrm{op}}$ its enveloping algebra.

Write $\mathcal{D}^{\mathrm{b}}_{e}(A) = \{X \in \mathcal{D}^{\mathrm{b}}(A) : \mathrm{H}^{*}(X) \in \mathrm{fd}(A/AeA)\}$.

Definition

A is internally bimodule 3-Calabi–Yau with respect to e if

(1) $A \in \operatorname{per} A^{\varepsilon}$ with $\operatorname{projdim}_{A^{\varepsilon}} A \leqslant 3$,

(2) there is a triangle

$$A \to \mathbf{R}\mathrm{Hom}_{A^{\varepsilon}}(A, A^{\varepsilon})[3] \to C \to A[1]$$

in $\mathcal{D}(A^{\varepsilon})$ such that $\mathbf{R}\mathrm{Hom}_{A}(C, M) = 0$ for all $M \in \mathcal{D}^{\mathrm{b}}_{e}(A)$, and $\mathbf{R}\mathrm{Hom}_{A^{\mathrm{op}}}(C, N) = 0$ for all $N \in \mathcal{D}^{\mathrm{b}}_{e}(A^{\mathrm{op}})$.

Consequence: $\operatorname{gl.dim} A \leqslant 3$ and $\mathrm{Ext}^{i}_{A}(X, Y) = \mathrm{D}\,\mathrm{Ext}^{3-i}_{A}(Y, X)$ for any $X \in \operatorname{mod} A$ and $Y \in \mathrm{fd}(A/AeA)$.

First main result

Theorem

Let D be a connected Postnikov diagram, with attached algebra A_D. Let e be the sum of idempotents given by the boundary (frozen) vertices. Then A_D is internally bimodule 3-Calabi–Yau with respect to e.

The proof uses the description of A_D as a frozen Jacobian algebra, and the following key observation, which is where the connectedness of D is used.

Lemma

Let D be a connected Postnikov diagram. Then A_D has a central subalgebra $Z \cong \mathbb{C}[[t]]$, and for each pair of vertices i and j, there is an isomorphism $e_j A e_i \cong Z$ of Z-modules.

It also follows from this lemma that A_D is Noetherian (because it is finitely generated over the commutative Noetherian ring Z) and that the quotient algebra $A_D/A_D e A_D$ is finite-dimensional, which we will use later.

Categorification

Theorem

Suppose A is a Noetherian $\mathbb{K}$-algebra and $e \in A$ an idempotent such that A is bimodule internally 3-Calabi–Yau with respect to e, and $\dim(A/AeA) < \infty$. Let $B = eAe$. Then

(1) *$B = eAe$ is Iwanaga–Gorenstein of Gorenstein dimension $\leqslant 3$; that is, B is Noetherian and $\operatorname{injdim}{}_B B, \operatorname{injdim} B_B \leqslant 3$. In particular,*

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}_B^{i>0}(X, B) = 0\}$$

is a Frobenius category.

(2) *the stable category $\underline{\mathrm{GP}}(B) = \mathrm{GP}(B)/\operatorname{proj} B$ is a 2-Calabi–Yau triangulated category.*

(3) *$A = \operatorname{End}_B(eA)^{\mathrm{op}}$ and $eA \in \mathrm{GP}(B)$ is cluster-tilting, that is*

$$\operatorname{add}(eA) = \{X \in \mathrm{GP}(B) : \operatorname{Ext}_B^1(X, eA) = 0\}.$$

Second main result

Theorem

Let D be a connected Postnikov diagram, with algebra A_D, let $e \in A_D$ be the boundary idempotent, and write $B_D = eA_De$. Then $\mathrm{GP}(B_D)$ is an additive categorification of the cluster algebra $\mathscr{A}_D \cong \mathbb{C}[\widehat{\Pi}^\circ(\sigma_D)]$.

This is just a corollary of the previous general result: A_D satisfies all of the assumptions by the first main result and its proof.

There is not a general definition of 'additive categorification': we use it here as shorthand to refer to the consequences of the previous general result, and many further consequences (e.g. concerning the mutation of cluster-tilting objects) due to many people.

One could (and should) ask for more: it is not yet proved that mutation of cluster-tilting objects in $\mathrm{GP}(B_D)$ is compatible with Fomin–Zelevinsky mutation of quivers, for example.

Boundary algebras

Since the cluster algebra $\mathscr{A}_D$, and the positroid variety $\Pi^\circ(\sigma_D)$, depend only on the permutation of σ_D, this should also be true of our category.

Proposition

If D and D' are connected Postnikov diagrams with $\sigma_D = \sigma_{D'}$, then $B_D \cong B_{D'}$, and so in particular $\mathrm{GP}(B_D) \simeq \mathrm{GP}(B_{D'})$.

This uses a result of Oh–Postnikov–Speyer; D and D' as in the Proposition are related by a sequence of local moves (which correspond to mutations of the quiver and in the cluster algebra!) which affect the isomorphism class of A_D, but not of the subalgebra $B_D = eA_De$.

The proof is really due to Baur–King–Marsh, who state the result for diagrams with $\sigma_D \colon i \mapsto i + k \mod n$.

The Jensen–King–Su category

Jensen–King–Su describe, for each $1 \leqslant k \leqslant n$, a Gorenstein algebra $B_{k,n}$ (directly, via a quiver with relations) such that $\mathrm{GP}(B_{k,n})$ categorifies the Grassmannian cluster algebra $\mathbb{C}[\widehat{\mathrm{Gr}}_k^n]$.

This cluster algebra is (up to inverting frozen variables) $\mathscr{A}_D$ in the case that D a Postnikov diagram with permutation $\sigma_D \colon i \mapsto i + k \mod n$ [Scott].

Theorem (Baur–King–Marsh)

If D has permutation $\sigma_D \colon i \mapsto i + k \mod n$, then $B_D \cong B_{k,n}$.

Thus we recover Jensen–King–Su's result as a special case, but via a different description of $B_{k,n}$.

Unlike in the general case, it is known that mutation of cluster-tilting objects in $\mathrm{GP}(B_{k,n})$ induces Fomin–Zelevinsky mutations of quivers.

It is also better understood how the objects of $\mathrm{GP}(B_{k,n})$ are related to functions on the corresponding positroid variety (which is dense in the Grassmannian Gr_k^n in this case).

The Jensen–King–Su category

We call a Postnikov diagram with n strands of 'average length' k a (k, n)-diagram. For example, if $\sigma_D \colon i \to i + k \mod n$ then D is a (k, n)-diagram, whose strands have constant length k.

Note: this is the k such that $\Pi^\circ(\sigma_D) \subseteq \mathrm{Gr}_k^n$.

Proposition (Çanakçı–King–P)

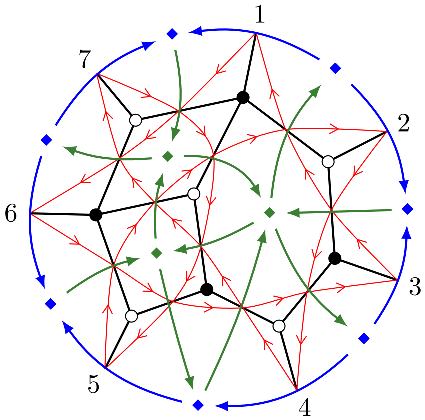
Let D be a (k, n)-diagram. Then there is a canonical ring morphism $B_{k,n} \to B_D$, inducing a fully-faithful functor $\mathrm{GP}(B_D) \to \mathrm{GP}(B_{k,n})$.

This means the categories we construct here all appear as full subcategories in Jensen–King–Su's Grassmannian cluster category, for the appropriate k and n.

Idea of proof: there is a canonical map $\Pi \to B_D$ for Π the preprojective algebra of type $\tilde{\mathsf{A}}_{n-1}$, since A_D is a frozen Jacobian algebra whose frozen subquiver is an orientation of this graph.

We check that the above canonical map factors over $B_{k,n}$, which is by definition a quotient of Π.

Thanks for listening!



Stay safe, and see you soon!