Fusion categories as (quantum) symmetries: stability conditions & Morita duality

@ FD Seminar 10 April 2024

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jt. work w H. Dell, A. Licata

G & rep(G)

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G-action and Morita duality

Let $\mathcal{A} =$ A-mod for A a f.d. algebra.

Suppose $\underbrace{G \circlearrowright A}_{\alpha :\, G \longrightarrow \mathrm{Aut}(A)}$ for G a finite group.

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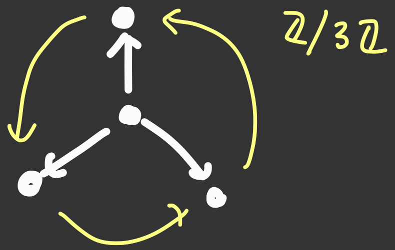
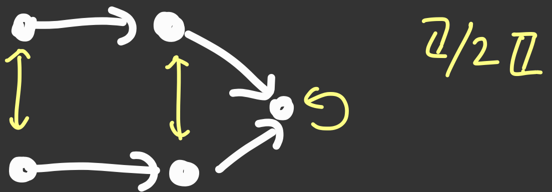
E.g. ① If $A = kQ$ for a quiver Q, then $G \subseteq \mathrm{Aut}(Q) \curvearrowright A$.





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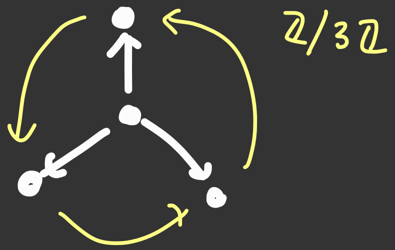
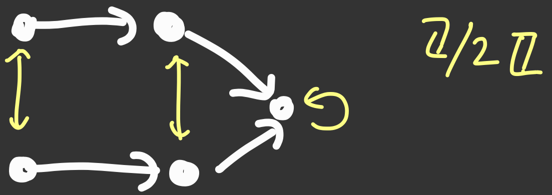
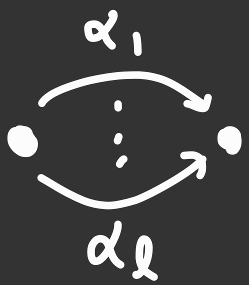




②

$Q_\ell :=$



For any finite G and a choice of $V \in \mathrm{rep}(G)$ $\underline{\text{w}}$ $\dim(V) = \ell$,

$$G \curvearrowright kQ_\ell$$

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Suppose $\underbrace{G \curvearrowright A}_{\alpha : G \to \mathrm{Aut}(A)}$ for G a finite group.

$$\rightsquigarrow \quad \mathcal{A}^G = \underbrace{(A\#G)}_{\substack{A\#G = A[G] \text{ as vector space} \\ ag \cdot bh = (a\alpha_g(b))(gh)}}\text{-mod}$$

Defn. $A\#G$ is called the skew group algebra.

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Let $\mathcal{A}$ = A-mod for A a f.d. algebra.

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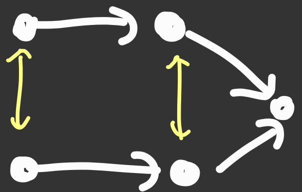
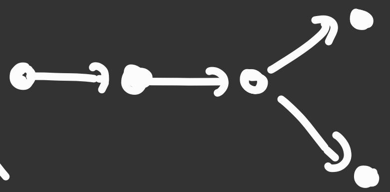
E.g. ① Q =  $\circlearrowleft \mathbb{Z}/2\mathbb{Z}$ $kQ\#G$-mod $\cong$ rep()

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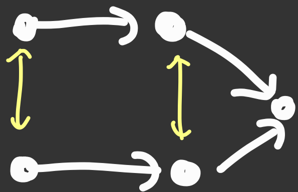
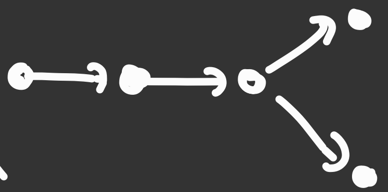
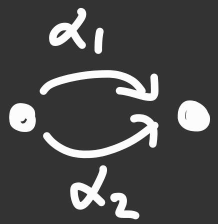
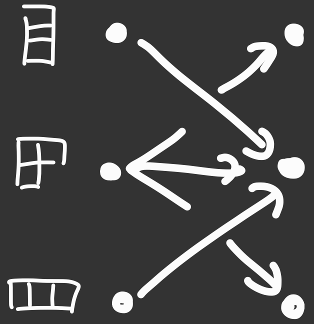
E.g. ① $Q =$  $\circlearrowleft \mathbb{Z}/2\mathbb{Z}$ $\quad kQ\#G\text{-mod} \cong \text{rep}\Big($  $\Big)$

② $Q_2 =$ 

$kQ_2\#G\text{-mod} \cong \text{rep}\Big($  $\Big)$

$G = S_3$ and  $\in \text{rep}(S_3)$

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A more "categorical" p.o.v.:

Defn. $\mathcal{A}^G$ is the abelian category w:

- objects: $(X, (\varphi_g))$ for $X \in \mathcal{A}$ and $\varphi_g : g \triangleright X \xrightarrow{\cong} X$

  s.t. $\varphi_g \circ \varphi_h = \varphi_{gh}$.
- morphisms: those in $\mathcal{A}$ commuting w iso. data.

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Given $V \in \mathrm{rep}(G)$, $M \in \mathcal{A}^G$

$$V \triangleright M := V \otimes_k M \text{ as vector space}$$

$$ag\cdot(v \otimes x) := (g\cdot v) \otimes (ag\cdot x).$$

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↳ This defines a monoidal functor $\mathrm{rep}(G) \longrightarrow \mathrm{End}_{ex}(\mathcal{A}^G)$

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BUT why care?

"G-invariant problems in $\mathcal{A}$"

$$\equiv$$

"rep(G)-equivariant problems in $\mathcal{A}^G$"

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Theorem [Macrì-Mehrotra-Stellari]

① Given $G \curvearrowright \mathcal{D}$,

$$\mathrm{Stab}_G(\mathcal{D}) \xhookrightarrow{\text{closed embedding}} \mathrm{Stab}(\mathcal{D}^G)$$

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Theorem [Macrí–Mehrotra–Stellari] [Perry–Pertusi–Zhao, Dell]

① Given $G \curvearrowright D$,

$$\mathrm{Stab}_G(D) \xhookrightarrow{\text{closed embedding}} \mathrm{Stab}(D^G)$$

② If G is abelian, then the image is $\mathrm{Stab}_{\hat{G}}(D^G)$, where $\hat{G} := \mathrm{Hom}(G, \Bbbk^\times)$

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Theorem [DHL]

Given $G \curvearrowright D$,

$$\boxed{\mathrm{Stab}_G(D) \cong \mathrm{Stab}_{\mathrm{rep}(G)}(D^G)}$$

SLOGAN: $\mathrm{rep}(G)$ captures all of the (quantum) symmetries dual to G (that was missing from just $\hat{G}$).

PLAN OF TALK

① Recap on stability conditions on abelian categories

↳ G-invariant and rep(G)-equivariant stability conditions

② Introduce the corresponding notions for Bridgeland stability conditions on triangulated categories.

③ The closed submfld. property and the duality

$$\mathrm{Stab}_G(\mathcal{D}) \cong \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}^G).$$

④ Example for ℓ-Kronecker quivers

⑤ Fusion categories & more...

Stability functions on abelian categories

Throughout, $\mathcal{A}$ is an abelian cat. (e.g. $\mathcal{A} =$ A-mod)

Defn. A stability function on $\mathcal{A}$ is a group morphism $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ s.t. $\forall E \neq 0 \in \mathcal{A}$,

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We say Z satisfies the **HN property** if $\forall\, X \in \mathcal{A}$,

$$0 \subseteq X_1 \subseteq X_2 \subsetneq \cdots \subsetneq X_n = X$$

with $E_i := X_i/X_{i-1}$ semistable and $\phi_i > \phi_{i+1}$.

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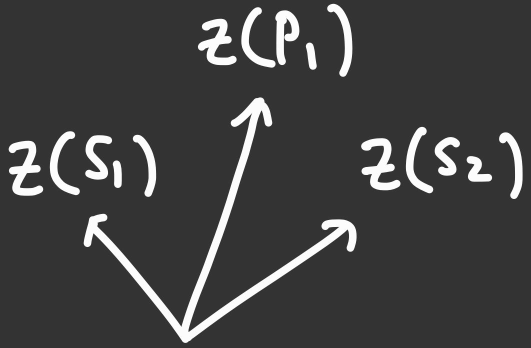

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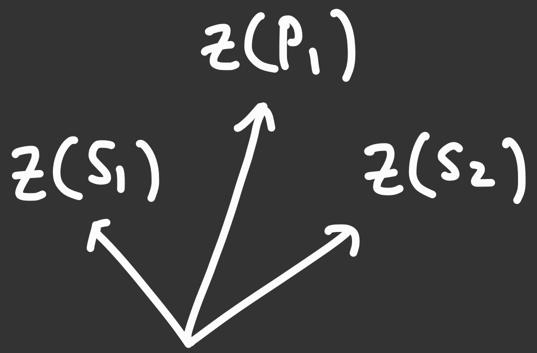

- simples are always semistable
- $S_2 \subseteq P_1$ and $\phi(S_2) < \phi(P_1)$ so P_1 is (semi)stable too.

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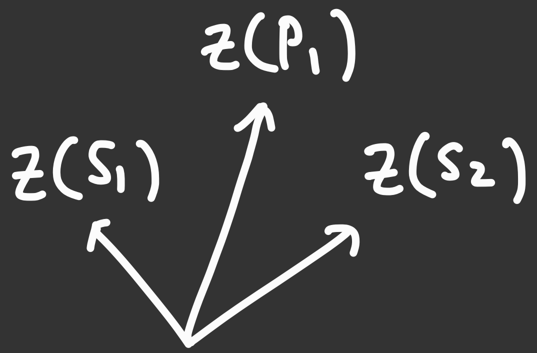

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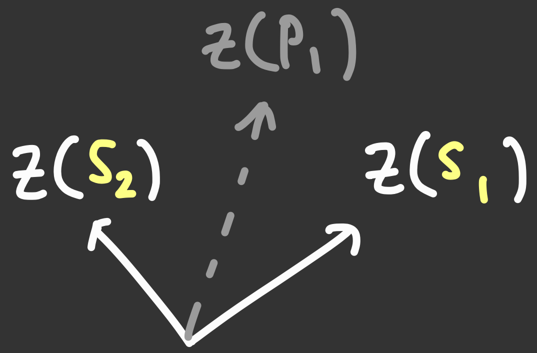

- P_1 is no longer semistable since $\phi(S_2) > \phi(P_1)$

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N.B. If E and F are both semistable w/ $\phi(E) > \phi(F)$, then $\mathrm{Hom}_{\mathcal{A}}(E, F) = 0$

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$$E \xrightarrow{f \neq 0} F$$

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$$\begin{tikzcd}
E \arrow[r, "f \neq 0"] \arrow[d, two heads] & F \\
\mathrm{coker}(f) \cong & \mathrm{im}(f) \arrow[u, hook]
\end{tikzcd}$$

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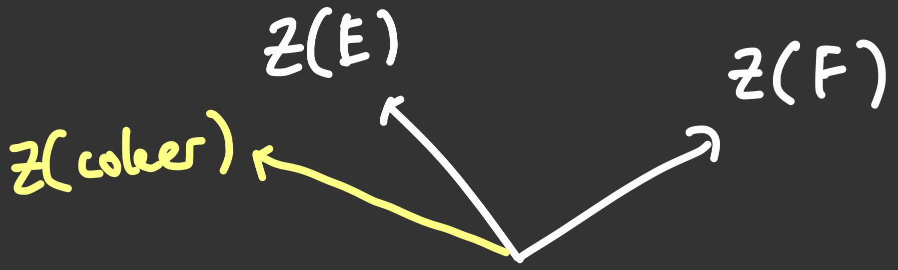

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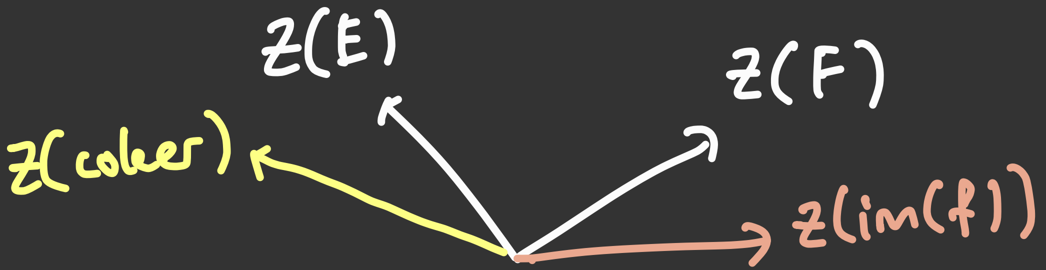

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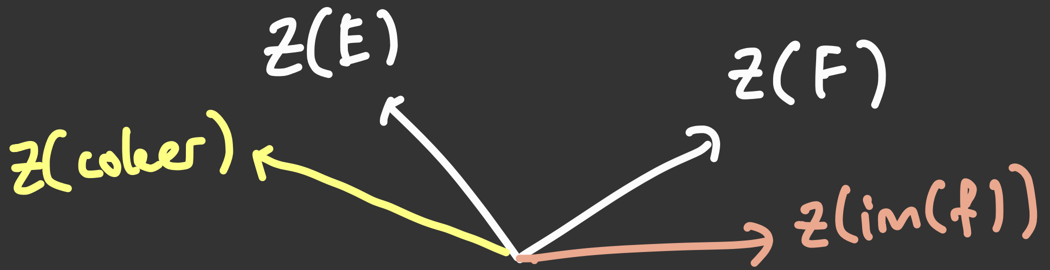

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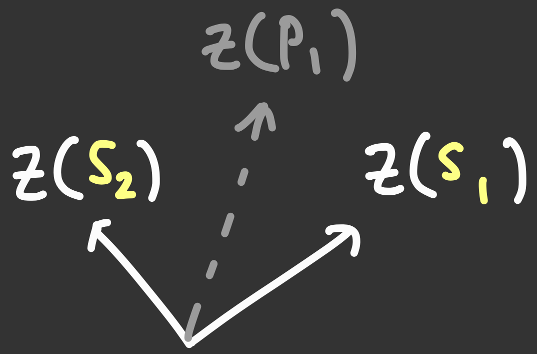

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Defn. A King's stability condition is a grp. homomorphism

$$\begin{tikzcd} \theta : K_0(A\text{-mod}) \arrow[r] & \mathbb{R} ; \\ \mathbb{Z}^n = \bigoplus \mathbb{Z}\{S_i\} \arrow[u, phantom, "\cong" rotate=90] & \end{tikzcd}$$

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Then M is v-semistable iff M is Z-semistable w/ phase $\frac{1}{2}$.

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Throughout, $\mathcal{A}$ is an abelian cat. with a G-action.

Defn. A stability function Z on $\mathcal{A}$ is G-invariant if $Z(g \triangleright E) = Z(E)$ for all $g \in G$, $E \in \mathcal{A}$.

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Defn. A stability function Z on $\mathcal{A}^G$ is $\mathrm{rep}(G)$-equivariant if $Z(V \triangleright M) = \dim(V) Z(M)$ for all $V \in \mathrm{rep}(G)$, $M \in \mathcal{A}^G$.

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Throughout, $\mathcal{A}$ is an abelian cat. with a G-action.

Defn. $\mathcal{P}(\phi) :=$ full subcategory of semistable objects with phase ϕ (N.B.: it is also abelian)

Prop. Suppose $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a G-invariant stability function. Then $E \in \mathcal{P}(\phi) \iff g \triangleright E \in \mathcal{P}(\phi) \ \forall g \in G$.

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Pf. If $X \subseteq g \triangleright E$, then $\underbrace{g^{-1} \triangleright X \subseteq E}_{\phi(g^{-1} \triangleright X) \leqslant \phi(E)}$.

Stability functions on abelian categories

Throughout, $\mathcal{A}$ is an abelian cat. with a G-action.

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Pf. If $X \subseteq g \rhd E$, then $\underbrace{g^{-1} \rhd X \subseteq E}$.

$$\begin{array}{ccc} \phi(g^{-1} \rhd X) & \leqslant & \phi(E) \\ = & & = \\ \phi(X) & & \phi(g \rhd E) \end{array}$$

$\square$

Stability functions on abelian categories

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Now $\mathcal{A}^G$ is also abelian, but has a $\mathrm{rep}(G)$-action.

Prop. [H] Suppose $Z : K_0(\mathcal{A}^G) \longrightarrow \mathbb{C}$ is a $\mathrm{rep}(G)$-equiv. stability function w the HN property.

Then $M \in \mathcal{P}(\phi) \iff V \triangleright M \in \mathcal{P}(\phi)$ $\forall V \in \mathrm{rep}(G)$.

The (1-cat Morita) duality for abelian categories

$$\begin{aligned} \mathrm{Forget} : \mathcal{A}^G &\longrightarrow \mathcal{A} \\ (X, (\varphi_g)) &\longmapsto X \end{aligned}$$

The (1-cat Morita) duality for abelian categories

$$\begin{tikzcd} \mathrm{Forget} : \mathcal{A}^G \arrow[r, shift left] & \mathcal{A} : \mathrm{Ind} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} (X, (\varphi_g)) \arrow[r, mapsto] & X \\ \left(\bigoplus_{g \in G} g \triangleright Y , \varepsilon_{nat} \right) & Y \arrow[l, mapsto] \end{tikzcd}$$

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$$\left\{\begin{matrix}\text{G-invariant stability} \\ \text{functions on } \mathcal{A}\end{matrix}\right\} \underset{\longleftarrow}{\overset{\longrightarrow}{\equiv}} \left\{\begin{matrix}\text{rep(G)-equivariant stability} \\ \text{functions on } \mathcal{A}^G\end{matrix}\right\}$$

$$Z \longmapsto Z \circ \mathrm{Forget} / |G|$$

$$W \circ \mathrm{Ind} \longleftarrow\!\shortmid\ W$$

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Moreover, $M \in \mathcal{A}^G$ is $(Z \circ \mathrm{Forget})$-semistable iff $\mathrm{Forget}(M) \in \mathcal{A}$ is Z-semistable (w the same phase).

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Moreover, $M \in \mathcal{A}^G$ is W-semistable iff $\mathrm{Forget}(M) \in \mathcal{A}$ is $(W \circ \mathrm{Ind})$-semistable ($\underline{w}$ the same phase).

And vice versa...

Bridgeland stability conditions

Let $\mathcal{D}$ be triangulated. (e.g. $\mathcal{D} = D^b(A\text{-mod})$)

Defn. A stability condition $\sigma = (Z, \mathcal{H})$ on $\mathcal{D}$ consists of

(i) a heart $\mathcal{H} \subseteq \mathcal{D}$ (which is abelian);

(ii) a stability function $Z : K_0(\mathcal{H}) \cong K_0(\mathcal{D}) \longrightarrow \mathbb{C}$

s.t. HN property holds.

Bridgeland stability conditions

Let D be triangulated. (e.g. $D = D^b(A\text{-mod})$)

Defn. A stability condition $\sigma = (Z, \mathcal{H})$ on D consists of

(i) a heart $\mathcal{H} \subseteq D$ (which is abelian);

(ii) a stability function $Z : K_0(\mathcal{H}) \cong K_0(D) \longrightarrow \mathbb{C}$

s.t. HN property holds.

Thm. [Bridgeland]

$$\begin{aligned} \mathrm{Stab}(D) &\longrightarrow \mathrm{Hom}(K_0(D), \mathbb{C}) \cong \mathbb{C}^n \\ (Z, \mathcal{H}) &\longmapsto Z \end{aligned}$$

is a local homeomorphism.

So $\mathrm{Stab}(D)$ is a f.d. complex manifold.

Bridgeland stability conditions

Let $\mathcal{D}$ be triangulated $\underline{w}$ a G-action. (e.g. $D^b(A\text{-mod})$)

Defn. A stability condition $\sigma = (Z, \mathcal{H})$ on $\mathcal{D}$ is G-invariant if (i) $\mathcal{H}$ is closed under the G-action;

(ii) Z is moreover G-invariant

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(e.g. $D^b(A\#G\text{-mod})$)

Now $\mathcal{D}^G$ has a rep(G)-action; supp. $\mathcal{D}^G$ is also triangulated.

Defn. A stability condition $\sigma = (Z, \mathcal{H})$ on $\mathcal{D}^G$ is rep(G)-equiv. if (i) $\mathcal{H}$ is closed under the rep(G)-action;

(ii) Z is moreover rep(G)-equivariant.

Bridgeland stability conditions

Let D be triangulated w/ a G-action. (e.g. $D^b(A\text{-mod})$)

Thm. [Macrì – Mehrotra – Stellari]

$\mathrm{Stab}_G(D) \longrightarrow \mathrm{Hom}(K_0(D), \mathbb{C})^G$ is a local homeomorphism.

Moreover, $\mathrm{Stab}_G(D) \subseteq \mathrm{Stab}(D)$ is closed.

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Let $\mathcal{D}$ be triangulated w a G-action. (e.g. $D^b(A\text{-mod})$)

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Moreover, $\mathrm{Stab}_G(\mathcal{D}) \subseteq \mathrm{Stab}(\mathcal{D})$ is closed.

Now $\mathcal{D}^G$ has a $\mathrm{rep}(G)$-action; supp. $\mathcal{D}^G$ is also triangulated.
(e.g. $D^b(A\text{-mod})^G$)

Thm. [DHL]

$$\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}) \longrightarrow \boxed{\mathrm{Hom}_{K_0(\mathrm{rep}(G))}(K_0(\mathcal{D}), \mathbb{C})}$$

← $\mathrm{rep}(G)$-equivariant central charges

is a local homeomorphism.

Moreover $\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}) \subseteq \mathrm{Stab}(\mathcal{D})$ is closed.

The (1-cat Morita) duality

$$\mathrm{Forget} : D^G \rightleftarrows D : \mathrm{Ind} \qquad \text{(biadjoints)}$$

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Defn. [Macrì - Mehrotra - Stellari]

Let $\Phi : \mathcal{D} \to \mathcal{D}'$. For $(P, Z) \in \mathrm{Stab}(\mathcal{D}')$, define

$$\Phi^{-1} \cdot (P, Z) := (\Phi^{-1} \cdot P, \; Z \circ \Phi)$$

where $(\Phi^{-1} \cdot P)(\phi) := \{ X \in \mathcal{D} \mid \Phi(X) \in P(\phi) \}$.

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★ not always a stability cond.!

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where $(\Phi^{-1} \cdot P)(\phi) := \{ X \in D \mid \Phi(X) \in P(\phi) \}$.

Thm. [DHL]

$$\mathrm{Forget}^{-1} : \mathrm{Stab}_G(D) \overset{\cong}{\rightleftarrows} \mathrm{Stab}_{\mathrm{rep}(G)}(D) : \mathrm{Ind}^{-1}$$

are mutually inverse homeomorphisms up to rescaling Z by $1/|G|$.

Kronecker and McKay quivers

Let

$$K_\ell := \begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_\ell"'] \arrow[r, phantom, "\vdots"] & 2 \end{tikzcd},$$

G a finite group.

Then each $V \cong \mathbb{C}^\ell \in \mathrm{rep}(G)$ defines an action of G on kK_ℓ, hence on $\mathrm{rep}(K_\ell)$.

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Claim. $\operatorname{rep}(K_\ell)^G \simeq \operatorname{rep}(Q_{G,V})$ where $Q_{G,V}$ is the *separated* McKay quiver.

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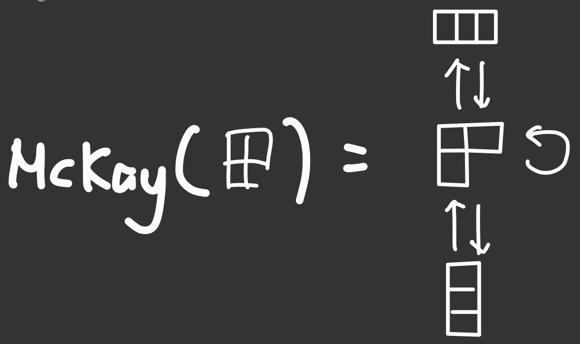
E.g. $G = S_3$.



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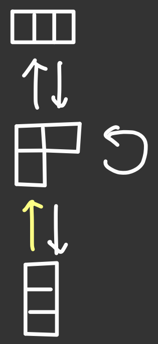
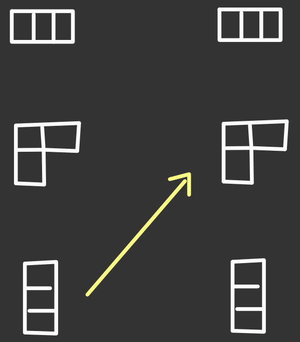
$\mathrm{McKay}($⊞$) =$



$Q_{G,}$⊞ $=$



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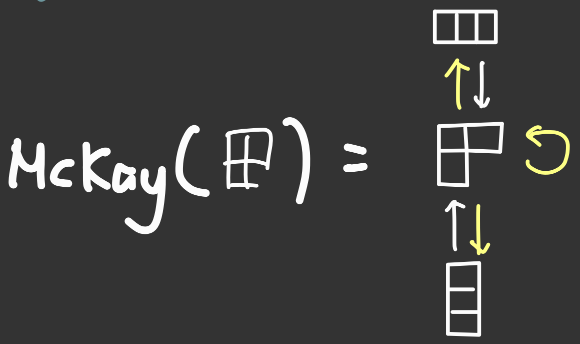

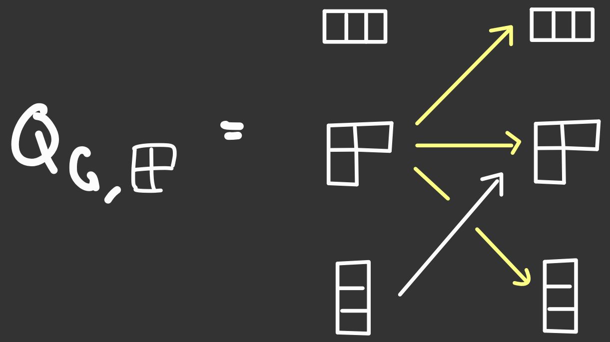

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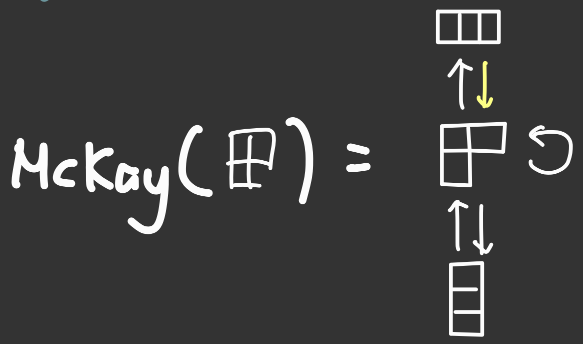

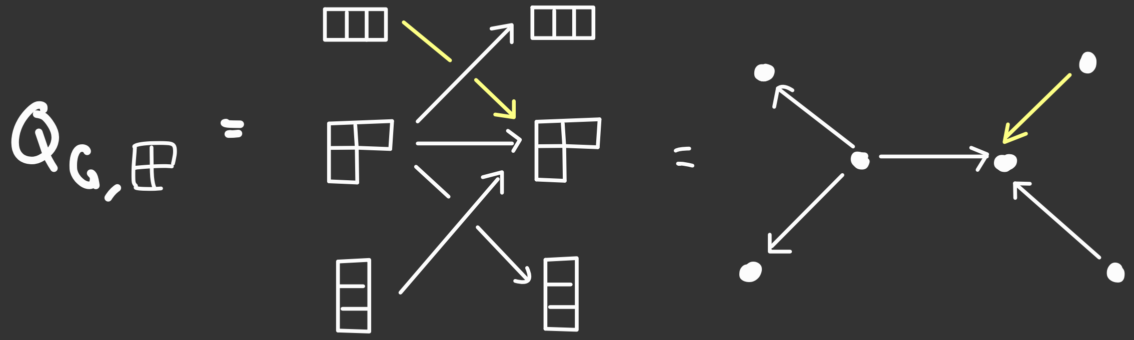

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N.B. Auslander–Reiten showed that $Q_{G,V}$ is of affine type iff $\dim(V) = 2$ (finite iff $\dim(V) = 1$).

Kronecker and McKay quivers

Corollary [DHL]

For each $V \in \operatorname{rep}(G)$ w $\ell = \dim(V)$,

$$\operatorname{Stab}_{\operatorname{rep} G}\left(D^b\operatorname{rep}(K_\ell)^G\right) = \operatorname{Stab}_{\operatorname{rep}(G)}\left(D^b\operatorname{rep}(Q_{G,V})\right)$$

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\mathrm{Stab}_G\left(D^b\mathrm{rep}(K_\ell)\right) \arrow[u, shift right=2, "\mathrm{Ind}^{-1}"'] & &
\end{tikzcd}$$

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\boxed{\mathrm{Stab}_G\left(D^b\mathrm{rep}(K_\ell)\right) = \mathrm{Stab}\left(D^b\mathrm{rep}(K_\ell)\right)} \arrow[u, shift right=2, "\mathrm{Ind}^{-1}"']
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\mathrm{Stab}_G\left(D^b\mathrm{rep}(K_\ell)\right) \arrow[u, shift right=2, "\mathrm{Ind}^{-1}"'] \arrow[r, phantom, "\overset{?}{=}"] \arrow[d] & \mathrm{Stab}\left(D^b\mathrm{rep}(K_\ell)\right) \arrow[d] \\
\mathrm{Hom}_G(K_0, \mathbb{C}) & \mathrm{Hom}(K_0, \mathbb{C})
\end{tikzcd}$$

Kronecker and McKay quivers

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since G fixes simples

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$\left.\begin{array}{l} \mathrm{Hom}_G(K_0, \mathbb{C}) = \mathrm{Hom}(K_0, \mathbb{C}) \\ \text{since } G \text{ fixes simples} \\ (\text{hence acts trivially on } K_0) \end{array}\right\}$ $\therefore$ Stab_G is a union of connected components

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\end{tikzcd}$$

$$\cong \begin{cases} \mathbb{C} \times \mathbb{H} & \ell \geq 3 \quad \text{[Dmitrov – Katzarkov]} \\ \mathbb{C}^2 & \ell = 1 \quad \text{[Haiden – K. – Kontsevich]} \\ & \ell = 2 \quad \text{[Okada]} \end{cases}$$

Other fusion categories

$\overline{\Bbbk} = \Bbbk$

Defn. A fusion category $\mathcal{C}$ is a semisimple, $\Bbbk$-linear category w monoidal structure $(\otimes, \mathbb{1})$ such that:

(i) $\mathbb{1}$ is simple;

(ii) every object has left & right duals;

(iii) finitely many simples.

E.g. ① $\mathcal{C} = \mathrm{rep}(G)$ $(\mathrm{char}(\Bbbk) \nmid |G|)$

② Fib $\longrightarrow$ two simples $\mathbb{1}, \pi$ and

$$\pi \otimes \pi \cong \mathbb{1} \oplus \pi$$

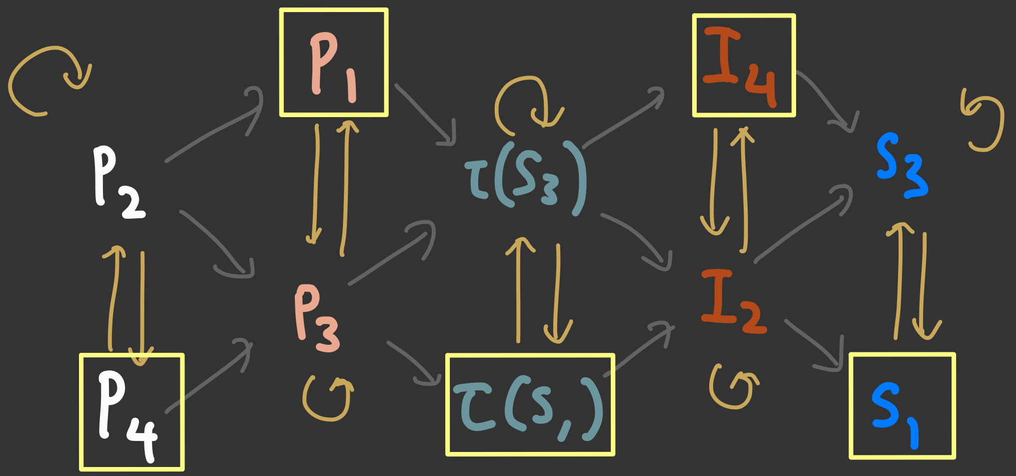
"Fibonacci / golden ratio category"

A golden-ratio symmetry: rep(A_4)

Algebra

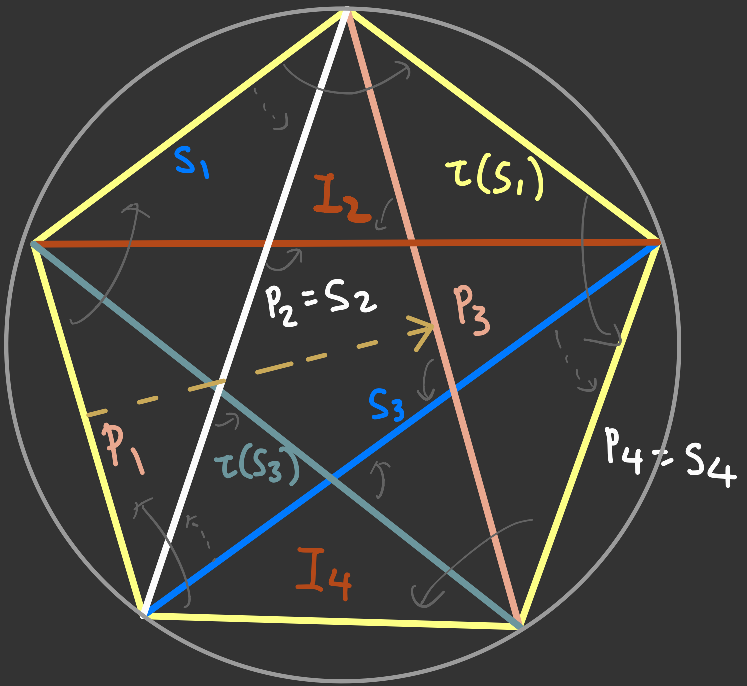
$$\mathrm{rep}\left(\begin{tikzcd}[row sep=small] & 4 \\ 3 \arrow[ur] \arrow[dr] & \\ & 2 \\ 1 \arrow[ur] & \end{tikzcd}\right) \circlearrowleft \mathrm{Fib} \ni \Pi$$

$$\Pi \otimes \Pi \cong \mathbb{1} \oplus \Pi$$



Geometry

$$\mathrm{DFuk}(\mathbb{D}_5) \overset{\circlearrowleft \mathrm{Fib}}{\cong} D^b(\mathrm{rep}(A_4))$$



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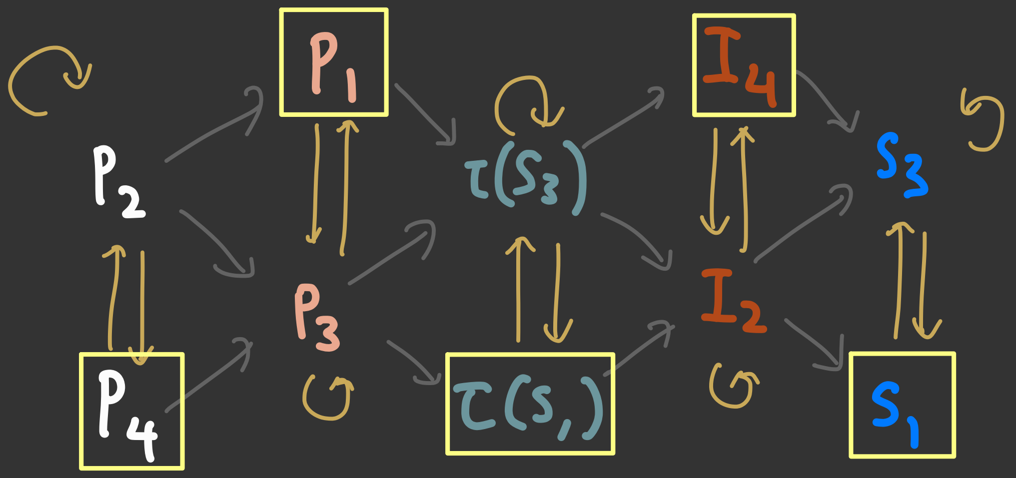

Geometry

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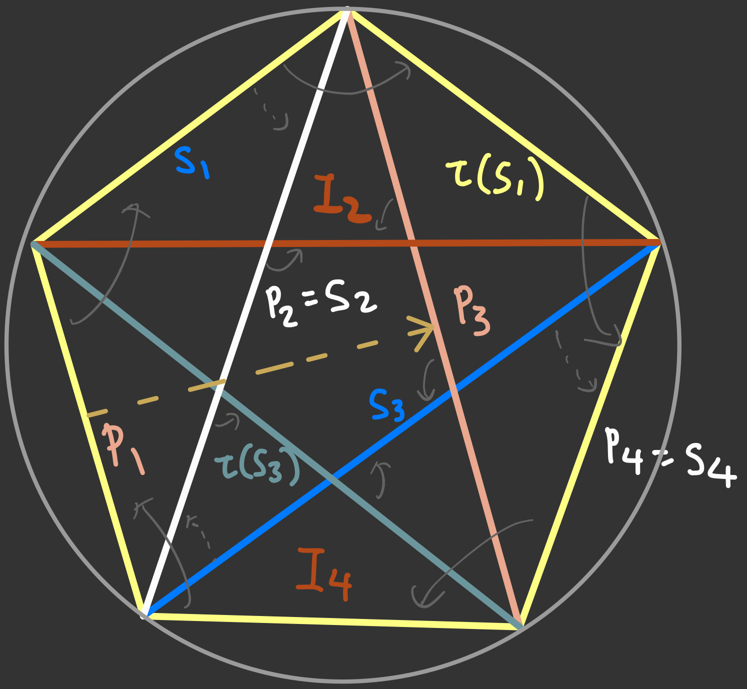

$\mathrm{Stab}_{\mathrm{Fib}}(\mathrm{DFuk}(\mathbb{D}_5))$ consists of metrics which gives "affine-transformed" regular pentagons.

A golden-ratio symmetry : Coxeter group $I_2(5)$

$$\mathcal{D} = \mathrm{Kom}\big(\underbrace{\mathrm{zig}(A_4)}\text{-prmod}\big) \circlearrowleft \mathrm{Fib}$$

$$\mathrm{zig}(A_4) := 1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \Big/ \left\langle \begin{array}{ll} i \curvearrowright j \curvearrowright k = 0 & i \neq k \\ \circlearrowright i = i \circlearrowleft & \end{array} \right\rangle$$

$$\begin{tikzcd} \mathrm{Stab}(\mathcal{D}) \arrow[d, "\text{universal cover}"] \\ \underbrace{\mathbb{C}^4 \setminus H_{A_4}}_{\text{A_4 hyperplane complement}} \end{tikzcd}$$

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$$\begin{tikzcd}
\mathrm{Stab}_{\mathrm{Fib}}(\mathcal{D}) \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathrm{Stab}(\mathcal{D}) \arrow[d, "\text{universal cover}"] \\
\underbrace{\mathbb{C}^2 \setminus H_{I_2(5)}}_{\text{pentagon hyperplane complement}} \arrow[r, phantom, "\subseteq"] & \underbrace{\mathbb{C}^4 \setminus H_{A_4}}_{A_4 \text{ hyperplane complement}}
\end{tikzcd}$$

$\mathbb{R}^2 \subseteq \mathbb{C}^2$

Thank You !