Semilinear clannish algebras

FD seminar

2204.12138, joint work with William Crawley-Boevey

Background: tame algebras

gentle — quadratic zero relations, not too many
Assem, Skowroński, '87

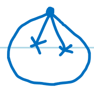
longer zero relations — string
Butler, Ringel, '87

PRECURSORS

skewed-gentle — quadratic zero relations, idempotent special loops
Geiß, Peña, '99

clannish
Crawley-Boevey, '89
longer zero relations, seperable special loops

Motivation:
- generalise clannish algebras to encompass interesting examples
- classify their finite-dimensional indecomposable modules

Extended/Dykin species of classical type	Clannish algebras with irreducibles	Existence of $\mathcal{F}$-crystals	Species from surfaces with orbifold points	more ..
Dlab, Ringel '76	c.f C-B '89	Kottwitz, Rapoport, '03	Geuenich, Labardini-Fragoso '17	
$\mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C} \xrightarrow{\mathbb{C}} \mathbb{C} \xrightarrow{\mathbb{H}} \mathbb{H}$	$\mathbb{R}\langle a,t\rangle/(a^2, t^2+1)$	${}_\sigma\mathbb{F}_p^n \rightleftarrows {}_\sigma\mathbb{F}_p^n$	 $\begin{tikzcd} \mathbb{C} \arrow[r, shift left] \arrow[r, shift right] & \mathbb{C} \arrow[dl] \\ \mathbb{R} \arrow[u] & \end{tikzcd}$	

§1: Semilinear path algebras

▫ denotes ingredient for semilinear clannish alg

▫ K, a division ring ← not necc. – commutative, e.g $\mathbb{H}$ – a f.g. $Z(K)$-module, e.g $K((x;\sigma))$

▫ $\mathrm{Aut}(K)$, group of ring automorphisms of K

A **K-ring** is a ring homomorphism $K \to R$ must be injective

So K-bimodule R with $\circ : R \times R \to R$, $1 \in R$ making a ring, and

$$\lambda(r \circ s) = (\lambda r) \circ s,\ (r \circ s)\lambda = r \circ (s\lambda),\ r \circ (\lambda s) = (r\lambda) \circ s,\ r1 = 1r$$

▫ $Q = \left(Q_0, Q_1, Q_1 \underset{t}{\overset{h}{\rightrightarrows}} Q_0\right)$, finite quiver

▫ $\underline{\sigma} = (\sigma_a : a \in Q_1)$, function $Q_1 \longrightarrow \mathrm{Aut}(K)$, $a \mapsto \sigma_a$

paths give right K-basis

Semilinear path algebra $K_{\underline{\sigma}}Q = \bigoplus_{p,\,\text{path}} Kp$, K-ring with

same as usual path algebra if K comm. and each $\sigma_a = 1$

generators: trivial paths e_v $(v \in Q_0)$, arrows $a \in Q_1$

relations: $e_u e_v = \begin{cases} e_v & (u = v) \\ 0 & (u \neq v) \end{cases}$ $\sum_v e_v = 1$, $a e_{t(a)} = a = e_{h(a)} a$, key difference ↓ $a\lambda = \sigma_a(\lambda) a$, $\lambda \in K$

skew-polynomials →

Example: $\sigma \in \mathrm{Aut}(K)$, $R = K[x;\sigma] \ni \sum_{i=0}^{n} \lambda_i x^i$ where $x\lambda = \sigma(\lambda)x$

Then $R \simeq K_{\underline{\sigma}}Q$ where Q is a loop x with $\sigma_x = \sigma$

§2: Semilinear representations

dually: V_σ for V, right K-module

For $\sigma \in \mathrm{Aut}(K)$ define twist ${}_\sigma V$ of left K-module V by restriction via σ.
Hence ${}_\sigma V$ is the same abelian group, where $\lambda \cdot V = \sigma(\lambda) v \quad (\lambda \in K, v \in V)$

hence $V \to W$ is σ-semilinear if $V \to {}_\sigma W$ K-linear

Call function $\theta: V \to W$ between K-modules σ-semilinear if

$$\theta(\lambda u + \mu v) = \sigma(\lambda)\theta(u) + \sigma(\mu)\theta(v) \quad \lambda, \mu \in K,\ u, v \in V$$

Example. $Q = 1 \xrightarrow{x} 2 \xrightarrow{y} 3$, $\sigma_x = \sigma$, $\sigma_y = \tau$. Then

generalised matrix ring

$$K_{\sigma,\tau} Q \simeq \begin{pmatrix} K & 0 & 0 \\ K_\sigma & K & 0 \\ K_{\tau\sigma} & K_\tau & K \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \nu & \eta & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \omega & 0 & 0 \end{pmatrix}$$

$\omega = \nu\tau(\sigma(\lambda)) + \eta\sigma(\mu) \iff$ coeff. of yx

---

In general: $K_{\underline{\sigma}} Q$ is the K-species given by K at each vertex and ${}_1K_{\sigma_a}$ at each arrow

${}_S W_S = K^{Q_1}$ where

$K_{\underline{\sigma}} Q \simeq T_S(W)$, $S = K^{Q_0}$ (tensor ring)

$(\lambda_v)(w_a)(\mu_v) = (\lambda_{h(a)} w_a \sigma_a(\mu_{t(a)}))$

- Gabriel, '72, '73
- Dlab, Ringel, '76

(reprentation type of species)

$(M, \theta) = (M_v, \theta_a : v \in Q_0, a \in Q_1)$ is a $\underline{\sigma}$-semilinear K-representation if

M_v is a left K-module $\forall v$, $\theta_a : M_{t(a)} \to M_{h(a)}$ is σ_a-semilinear $\forall a$

- Jacobson, '35
- Ringel, '05

(semilinear representations)

$f = (f_v : M_v \to N_v)$ is a morphism $(M, \theta) \to (N, \varphi)$ of such objects if

f_v is K-linear $\forall v$ and $\varphi_a f_{t(a)} = f_{h(a)} \theta_a \ \forall a$

§3: Semilinear clannish algebras

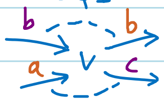
- $\mathcal{Z}$, some subset of zero relations paths, length ≥ 2
- $\mathcal{S}$, some subset of special loops $\mathcal{S} \subseteq \{s \in Q_1 : h(s) = t(s)\}$

[Crawley-Boevey, '89] Call $(Q, \mathcal{S}, \mathcal{Z})$ clannish if

$Q = a \circlearrowleft v \circlearrowright s$
$\mathcal{S} = \{s\}$
$\mathcal{Z} = \{a^2\}$

(1) $p \in \mathcal{Z},\ s \in \mathcal{S} \Rightarrow p \neq rs, sq, rs^2q$ (paths r, q)

$b \notin \mathcal{S}$



(2) $v \in Q_0 \Rightarrow \begin{cases} \text{at most two } a \in Q_1 : h(a) = v \\ \text{at most two } c \in Q_1 : t(c) = v \end{cases}$

can instead take $\mathcal{Z} = \{a^2, asa\}, \{a^2, asasa\}, \ldots$

(3) $b \in Q_1, b \notin \mathcal{S} \Rightarrow \begin{cases} \text{at most one } a \in Q_1 : h(a) = t(b) \text{ and } ba \notin \mathcal{Z} \\ \text{at most one } c \in Q_1 : t(c) = h(b) \text{ and } cb \notin \mathcal{Z} \end{cases}$

b 'ordinary'

---

last ingredient
- For each $s \in \mathcal{S}$ at $v \in Q_0$, choose $q_s(x) = x^2 - \beta_s x + \gamma_s \in K[x; \sigma_s]$

Then let
$$q(s) = s^2 - \beta_s s + \gamma_s e_v \in e_v K_\sigma Q e_v$$

so choose $(\beta_s, \gamma_s) \in K^2$

Definition. [BT, Crawley-Boevey, '22] Semilinear clannish algebras are K-rings of the form

$$K_\sigma Q / \langle \mathcal{Z} \cup \{q(s) : s \in \mathcal{S}\} \rangle$$

where

$(Q, \mathcal{S}, \mathcal{Z})$ clannish, $\sigma \in \mathrm{Aut}(K)^{Q_1}$ and $q_s(x) = x^2 - \beta_s x + \gamma_s \in K[x; \sigma_s]$ $(s \in \mathcal{S})$

For module classification later: mild conditions imposed on each $q_s(x) = x^2 - \beta_s x + \gamma_s \in K[x; \sigma_s]$

§4: Examples

precursors, K comm.:

- string alg, $\mathcal{S}=\emptyset$, $\sigma_a=1$ (Butler, Ringel, '87)
- clannish alg, $\sigma_a=1$, $q_s(x)$ not irreducible (Crawley-Boevey, '89)

Dynkin species: [Dlab, Ringel, '76]

$K=\mathbb{C}$, $Q = \overset{s}{\circlearrowleft} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \overset{t}{\circlearrowright}$, $\mathcal{S}=\{s,t\}$, $Z=\emptyset$,

$\xi\in \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ conjugation, $\sigma_a=\sigma_b=\mathrm{id}_{\mathbb{C}}$, $\sigma_s=\sigma_t=\xi$,

$q_s(x)=x^2-1$, $q_t(x)=x^2+1$.

$$\left.\begin{array}{l}\mathbb{C}[x,\sigma_s]/\langle q_s(x)\rangle \cong M_2(\mathbb{R})\\ \mathbb{C}[x,\sigma_t]/\langle q_t(x)\rangle \cong \mathbb{H}\end{array}\right\} \Rightarrow \mathbb{C}_\sigma Q/\langle s^2-e_1, t^2+e_2\rangle \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & 0 & 0\\ \mathbb{R} & \mathbb{R} & 0 & 0\\ \mathbb{C} & \mathbb{C} & \mathbb{C} & 0\\ \mathbb{H} & \mathbb{H} & \mathbb{H} & \mathbb{H}\end{pmatrix}$$

$$\overset{M}{\sim} \begin{pmatrix} \mathbb{R} & 0 & 0\\ \mathbb{C} & \mathbb{C} & 0\\ \mathbb{H} & \mathbb{H} & \mathbb{H}\end{pmatrix}$$

tensor ring of species $\mathbb{R} \xrightarrow{\mathbb{C}} \mathbb{C} \xrightarrow{\mathbb{H}} \mathbb{H}$ of type $\widetilde{BC}_2$: $\bullet \xrightarrow{(1,2)} \bullet \xrightarrow{(1,2)} \bullet$

A_4 representations: [Dlab, Ringel, '89]

$K=\mathbb{F}_4=\mathbb{F}_2(\omega)$, $\omega^2+\omega+1=0$, $\tau\in \mathrm{Aut}_{\mathbb{F}_2}(\mathbb{F}_4)$, $\tau(\omega)=\omega^2$

$Q = \overset{c}{\circlearrowleft} 1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \overset{s}{\circlearrowright}$ $\quad\sigma_a=\mathrm{id}_{\mathbb{F}_4}$, $\sigma_b=\sigma_c=\sigma_s=\tau$

$\mathcal{S}=\{s\}$, $q_s(x)=x^2-1\in \mathbb{F}_4[x;\tau]$, $Z=\{ab, ac, ba, cb\}$

$$\mathbb{F}_{4\,\sigma} Q/\langle ab, ac, ba, cb, s^2-e_2\rangle \overset{M}{\simeq} \mathbb{F}_2 A_4/\mathrm{soc}(\mathbb{F}_2 A_4)$$

A_4, alternating group

$\mathbb{F}_2 A_4$ symmetric, so $\mathrm{soc}(\mathbb{F}_2 A_4)$ annihilates non-proj. ind. f.d rep's

Dedekind-like: [Klingler, Levy, '01]

$K=\mathbb{Q}(i)$, $\xi \in \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(i))$ conjugation, $Q = s \circlearrowright 1 \circlearrowleft a$

$\sigma_a = \sigma_s = \xi$, $q_s(x) = x^2-1$

$$\mathbb{Q}(i)_\sigma Q / \langle a^2, s^2 - e_1\rangle \simeq M_2(R) \underset{M}{\simeq} R$$

where $R = \mathbb{Q} + x\,\mathbb{Q}(i)[x]$, subring of $\mathbb{Q}(i)[x]$ Dedekind domain

Dedekind-like ring, unsplit type

On $\mathcal{F}$-crystals: [Kottwitz, Rapoport, '03]

$K=\overline{\mathbb{F}_p}$ ($p>0$ prime), $\varphi\in \mathrm{Aut}_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$, $\varphi(z)=z^p$ Frobenius automorphism

$$\begin{tikzcd}
1 \arrow[rr, bend left, "a_1"] \arrow[dr, bend left, "c_3"] & & 2 \arrow[ll, bend left, "c_1"] \arrow[dl, bend left, "a_2"] \\
& 3 \arrow[ul, bend left, "a_3"] \arrow[ur, bend left, "c_2"] &
\end{tikzcd}$$

$\sigma_{a_i}, \sigma_{b_i} \in \langle\varphi\rangle \subseteq \mathrm{Aut}_{\mathbb{F}_p}(\overline{\mathbb{F}_p})$ $\langle\varphi\rangle = \{\varphi^n \mid n\in\mathbb{Z}\}$

$\mathcal{S} = \emptyset$, $Z = \{a_i b_i, b_i a_i \mid i=1,2,3\}$

σ-semilinear $\overline{\mathbb{F}_p}$-representations of Q, bounded by Z,
related to arithmetic geometry [Ringel, '05]   lecture unpublished

---

Prototypical: $\dfrac{K_{\rho,\tau}\langle a,t\rangle}{(a^2, t^2-\beta t+\gamma)}$

K arbitrary division ring, $Q = a \circlearrowright v \circlearrowleft t$, $\mathcal{S}=\{t\}$, $Z=\{a^2\}$

$\sigma_a=\rho$, $\sigma_t=\tau \in \mathrm{Aut}(K)$ arbitrary, $q_t(x) = x^2-\beta x+\gamma \in K[x;\tau]$ with mild restrictions

§5: Modules

Four types: asymmetric strings, symmetric strings, asymmetric bands, symmetric bands

Aim: describe modules for $K_{\rho,\tau}\langle a,t\rangle/(a^2, t^2-\beta t+\gamma)$, prototypical

K, ρ, τ arbitrary
mild conditions on β, γ

Word: alternating sequence between (a or a^{-1}) and $t^\bullet$

later choose $\bullet = \pm 1$

String: finite word whose first and last letters are both $t^\bullet$

asymmetric string $w = t^\bullet a^{-1} t^\bullet a\, t^\bullet a^{-1} t^\bullet a\, t^\bullet a\, t^\bullet$

$0 \quad 1 \quad 2 \quad 3 \quad \cdots \quad 10 \quad 11$

Band: left-right-infinite word which is periodic

asymmetric band $w = \ldots t^\bullet a^{-1} t^\bullet a\, t^\bullet a\, t^\bullet a^{-1} t^\bullet a\, t^\bullet a\, t^\bullet a^{-1} t^\bullet a \ldots$

$\cdots \; -2 \; -1 \; 0 \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; \cdots$

Symmetric string/band: reflection symmetry/symmetries about $t^\bullet$

symmetric string $w = t^\bullet a^{-1} t^\bullet a\, t^\bullet a^{-1} t^\bullet a\, t^\bullet$

or

symmetric band $w = \ldots a\, t^\bullet a\, t^\bullet a^{-1} t^\bullet a^{-1} t^\bullet a\, t^\bullet a\, t^\bullet a^{-1} t^\bullet a^{-1} t^\bullet a\, t^\bullet \ldots$

Orientation. Walk of string/band w found by replacing each t° with t or t^{-1}, according to direction of nearest a's which break reflection symmetry

formal definition defined using an order, given by considering $\mathbb{Z}$

Example: given $w = t^\circ a^{-1} t^\circ a\, t^\circ a^{-1} t^\circ a\, t^\circ a\, t^\circ$

The walk is $t^{\pm 1} a^{-1} t^{\pm 1} a\, t^{\pm 1} a^{-1} t^{\pm 1} a\, t^{\pm 1} a\, t^{\pm 1}$

Each walk defines a quiver Q^w (built with $\leftarrow\ \leftarrow$) of type A_n (string) or ${}_\infty A_\infty$ (band)

Define left module M_w over $R = K_{\beta,\gamma}\langle a,t\rangle/(a^2, t^2-\beta t+\gamma)$

generators: $b_i \quad (i \in Q_0^w)$

relations: $ab_i = 0 \quad (i-1 \notin Q_0^w$ or $i+1 \notin Q_0^w)$

$ab_i = b_j \quad (i \xrightarrow{a} j \in Q_1^w), \quad t b_i = b_j \quad (i \xrightarrow{t} j \in Q_1^w)$

$(b_i : i \in Q_0^w)$ is a left K-basis for M_w

Consequently: $ab_j = 0 \quad (i \xrightarrow{a} j \in Q_1^w), \quad t b_j = \beta b_j - \gamma b_i \quad (i \xrightarrow{t} j \in Q_1^w)$

$ab_j = a(ab_i) \ldots \qquad t b_j = t(t b_i) \ldots$

For w a string or band, define parameterising ring

semisimple

σ defined by w

free product over K

$$A_w = \begin{cases} K & \text{(asymmetric string)} \\ K[x;\tau]/(q(x)) & \text{(symmetric string)} \\ K[x,x^{-1};\sigma] & \text{(asymmetric band)} \\ K[x,\tau]/(q(x)) \ast_K K[y;\tau]/(q(y)) & \text{(symmetric band)} \end{cases}$$

For each w above we show M_w is an R-A_w-bimodule

$R = K_{\rho,\tau}\langle a,t\rangle/(a^2, t^2-\beta t+\gamma)$

First step: show M_w is R-K-bimodule, so want $b_i\lambda \in M_w$ $(\lambda \in K, i \in Q_0^w)$ such that $c(b_i\lambda) = (cb_i)\lambda$ for $c = a,t$

choice of π_i's doesn't matter to the R-module $M_w \otimes_{A_w} V$

Trick: let $b_i\lambda := \pi_i(\lambda)b_i$ and solve for $\pi_i \in \mathrm{Aut}(K)$

Given $i \xrightarrow{a} j \in Q_1^w$, $a\pi_i(\lambda)b_i = \rho(\pi_i(\lambda))b_j$, so want

$$\rho\pi_i = \pi_j \quad (i \xrightarrow{a} j) \qquad \tau\pi_i = \pi_j \quad (i \xrightarrow{t} j)$$

Example: $w = t^{\bullet} a^{-1} t^{\bullet}$, $Q^w = 0 \xrightarrow{t} 1 \xrightarrow{a} 2 \xrightarrow{t} 3$, let $\pi_0 = \mathrm{id}_K$, then

$\pi_1 = \rho\pi_0 = \rho$, $\pi_2 = \tau\pi_1 = \tau\rho$, $\pi_3 = \rho\tau\rho$

V, K-module.
$\lambda\cdot v = \sigma(\lambda)v$ in ${}_\sigma V$

Want to consider $M_w \otimes_{A_w} V$, V indecomposable A_w-module

$\lambda b_i \otimes \mu = b_i \otimes \pi_i^{-1}(\lambda)\mu$

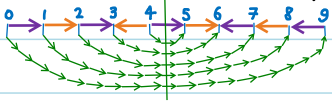
Asymmetric strings: $A_w = K$. Note $b_i \otimes K \simeq {}_{\pi_i^{-1}}K$, combine to draw $M_w \otimes_K K$

For w with $Q^w = 0 \to 1 \to 2 \leftarrow 3 \leftarrow 4 \leftarrow 5 \leftarrow 6 \leftarrow 7$ this gives

$${}_{\pi_0}K \to {}_{\pi_1}K \to {}_{\pi_2}K \leftarrow {}_{\pi_3}K \leftarrow {}_{\pi_4}K \leftarrow {}_{\pi_5}K \leftarrow {}_{\pi_6}K \leftarrow {}_{\pi_7}K$$

$\pi_0 = 1$, $\pi_1 = \tau^{-1}$, $\pi_2 = \tau^{-1}\rho^{-1}$, ..., $\pi_7 = \tau^{-1}\rho^{-1}\tau\rho\tau\rho\tau$

Symmetric strings: $A_w = K[x;\tau]/(x^2-\beta x+\gamma)$, x acts via reflection



$$b_i x = \begin{cases} b_{9-i} & (i \le 4) \\ b_i\beta - b_{9-i}\gamma & (i > 4) \end{cases}$$

by construction $(x-t)b_4 = 0$

$\rho = \tau = 1$ gives usual diagram

$$\begin{tikzcd} {}_{\rho^{-1}\tau\rho\tau}V \arrow[r] & {}_{\rho^{-1}\tau\rho}V \arrow[r] & {}_{\rho^{-1}\tau}V \arrow[r] & {}_{\rho^{-1}}V & V \arrow[l] \arrow[loop right, "x"] \end{tikzcd}$$

Asymmetric bands: $A_w = K[x, x^{-1}, \sigma]$, x acts via shift symmetry

$$\begin{tikzcd} V & {}_{\tau}V \arrow[l] & {}_{\tau\rho}V \arrow[l] & {}_{\tau\rho\tau}V \arrow[l] & {}_{\tau\rho\tau\rho}V \arrow[l] & {}_{\tau\rho\tau\rho\tau}V \arrow[l] \arrow[lllll, bend left, "x^{-1}"] \end{tikzcd}$$

$\sigma = \tau\rho\tau\rho\tau\rho^{-1}$

Symmetric bands: $A_w = K[x;\tau]/(q(x)) \ast_K K[y;\tau]/(q(y))$

$$\begin{tikzcd} {}_{\rho^{-1}}V \arrow[loop left, "y"] & V \arrow[l] & {}_{\tau}V \arrow[l] & {}_{\tau\rho}V \arrow[l] \arrow[loop right, "x"] \end{tikzcd}$$

Headline: for these new algebras, string and band modules are twisted according to semilinearity

Terminology: For the main theorem: given $\sigma \in \mathrm{Aut}(K)$ we say that a quadratic $q(x) = x^2 - \beta x + \gamma \in K[x;\sigma]$ is
i) normal if $q(x)P = Pq(x)$ where $P = K[x;\sigma]$
ii) non-singular if $\gamma \neq 0$
iii) semisimple if the artinian ring $P/(q(x))$ is semisimple

Notation: K-ring R, $\mathrm{ind}(R)$ denotes complete set of non-isomorphic indecomposable left R-modules M such that $\dim_K(M) < \infty$

Theorem: [BT, Crawley-Boevey, '22] Let $R = K_{\underline{\sigma}}Q/(Z \cup \{q(s) : s \in \mathbb{S}\})$ be semilinear clannish
Assume (i), (ii) and (iii) hold for each $q_s(x) \in K[x;\sigma_s]$

(i), (ii), (iii) mild up to basis change

Let W be a set of representatives of equivalence classes of strings and bands $w' \sim w$ iff $w' = w[n]$ or $w' = w^{-1}[n]$

As w runs through W and V_w runs through $\mathrm{ind}(A_w)$ the modules $M_w \otimes_{A_w} V_w$ run through $\mathrm{ind}(R)$

Things I like...

...Theorem.

symmetric band w, A_w is HNP, free product of s.s, nice K-basis, f.g / $K[x,x^{-1};\sigma]$

- Modules: after twisting look similar as rep.s of clans
- Parameters A_w: as above; in principal, modules understood
- Statement: direct generalisation; understood decompositions

...Proof.

consider tensor hom adjunction ${}_R M_w {}_{A_w}$

- Functorial filtrations: new potential; K-rings, semilinearity
- Spitting/Orientation lemmas: generalised, simplified
- New ideas involving functors $\mathrm{Hom}_R(M_w, -)$

... Context.

regular rep. of $K[x;\sigma]/\langle q(x)\rangle \leftrightarrow \begin{pmatrix} 0 & 1 \\ -\gamma & \beta \end{pmatrix}$ for $q(x) = x^2 - \beta x + \gamma$ irreducible

- Removed restrictions on clans: understood irreducible quadratics
- Encompasses previous considerations: introduction (above)
- Generality prompts questions (below)

Questions:

- Rings morita equiv. to semilinear dannish? Their modules?
- Applications; Ringel lecture? Others?
- Homological properties (gentle relations)? Homotopy words?
- Generalisations involving derivations? $K[x;\sigma,\delta]$?
- Infinite-dimensional representations? Purity?

Thank-you for your attention!