Locally free Caldero-Chapoton functions

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Rank 2 cluster algebras

Let b, c be two positive integers.

$$x_{n-1} \cdot x_{n+1} = \begin{cases} 1 + x_n^b & n \quad \text{odd} \\ 1 + x_n^c & n \quad \text{even} \end{cases}$$

$\mathcal{A}(b,c) \subset \mathbb{Q}(x_1, x_2)$ subalgebra generated by

$\{ x_n \mid n \in \mathbb{Z} \}$ (cluster variables).

Finite type example

$B_2/C_2 \qquad b=1,\ c=2$

$$\begin{tikzcd}[column sep=small]
 & x_1 \arrow[r, dash] \arrow[dl, dash] & x_2 \arrow[dr, dash] & \\
x_6 = x_4^{-1}(1+x_5) = x_2^{-1}(1+x_1) = x_0 \arrow[dr, dash] & & & x_3 = x_1^{-1}(1+x_2^2) \arrow[dl, dash] \\
 & x_1^{-1}x_2^{-2}(1+2x_1+x_1^2+x_2^2) = x_5 \arrow[r, dash] & x_4 = x_2^{-1}(1+x_3) = x_1^{-1}x_2^{-1}(1+x_1+x_2^2) &
\end{tikzcd}$$

Two f.d. algebras associated to (b, c)

$$\begin{tikzcd} Q: & 1 \arrow[loop above, "\varepsilon_1"] \arrow[r] & 2 \arrow[loop above, "\varepsilon_2"] \end{tikzcd}$$

$$H := \mathbb{C}Q / \langle \varepsilon_1^{c}, \varepsilon_2^{b} \rangle$$

path algebra

Geiss – Leclerc – Schröer

$g = \gcd(b, c)$, $\quad c_1 b = c_2 c$

$$\begin{tikzcd} Q: & 1 \arrow[loop left, "\varepsilon_1"] \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_g"'] \arrow[r, phantom, "\vdots"] & 2 \arrow[loop right, "\varepsilon_2"] \end{tikzcd}$$

$$I := \left\langle \varepsilon_1^{c_1}, \varepsilon_2^{c_2}, \varepsilon_2^{b/g} \alpha_k - \alpha_k \varepsilon_1^{c/g} \,\middle|\, 1 \leq k \leq g \right\rangle$$

$$H := \mathbb{C}Q / I$$

Def. $M \in \operatorname{mod} H$ is called locally free if

$e_i M$ is free over $e_i H e_i \cong \mathbb{C}[\varepsilon_i]/(\varepsilon_i^{c_i}) =: H_i$

$e_i M \cong H_i^{\oplus m_i}$ $\qquad$ (m_1, m_2) : the rank vector of M.

$Gr^{l.f.}(\underline{r}, M) := \{$ l.f. submodules of M with rank $\underline{r} = (r_1, r_2)\}$

(quasi-projective) subvariety of usual quiver grassmannian.

$\chi(\underline{r}, M) :=$ Euler characteristic of $Gr^{l.f.}(\underline{r}, M)$.

<u>Def</u>. (GLS) $M \in \operatorname{mod} H$ locally free. The l.f. Caldero–Chapoton function is

$$X_M := x_1^{-m_1} x_2^{-m_2} \sum_{\underline{r}=(r_1,r_2)} \chi(\underline{r}, M)\, x_1^{b(m_2-r_2)}\, x_2^{c r_1} \in \mathbb{Z}[x_1^{\pm}, x_2^{\pm}]$$

e.g. $E_1 := \mathbb{C}[\varepsilon_1]/(\varepsilon_1^c) \longrightarrow 0$ $\quad X_{E_1} = x_1^{-1}(1 + x_2^c) = X_3$

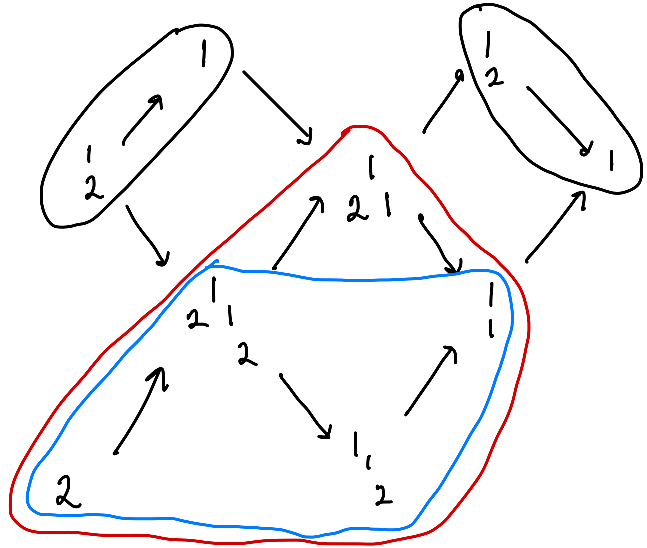
$E_2 := 0 \longrightarrow \mathbb{C}[\varepsilon_2]/(\varepsilon_2^b)$ $\quad X_{E_2} = x_2^{-1}(1 + x_1^b) = X_0$

e.g. $b=1$, $c=2$

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon_1^2=0"] \arrow[r] & 2 \end{tikzcd}$$

finite rep. type

AR-quiver



$$X_{P_1} = X_1^{-1}X_2^{-2}\left(X_1^2 + \chi(\mathbb{P}^1_c)\, X_1 + 1 + X_2^2 \right) = X_5$$

$$X_{I_2} = X_1^{-1}X_2^{-1}\left(X_1 + 1 + X_2^2 \right) = X_4$$

Thm (M) For $bc \geq 4$, we have bijection

$$\left\{ \text{rigid}_{\wedge} \text{ l.f. } H\text{-modules} \right\} \Big/ \sim \quad \overset{\sim}{\longleftrightarrow} \quad \left\{ X_n \;\middle|\; n \leq 0 \text{ or } n \geq 3 \right\}$$

indecomposable

$$M \longmapsto X_M(x_1, x_2)$$

Rmk. The bijection is natural (by GLS) from τ-tilting theory.

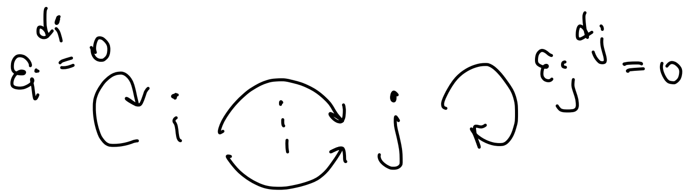
We show $X_{M(n)} = X_n$ where

$M(n)$ obtained by BGP-type reflections.

Beyond rank 2

GLS have associated $H(B,D)$ to any acyclic skew-symmetrizable B and D s.t. $DB + B^T D = 0$.

"Concatenate" rank 2 algebras



$g_{ij} = \gcd(b_{ij}, b_{ji})$ (with relations)

(when $b_{ij} < 0$)

X_M can be defined for locally free $M \in \operatorname{mod} H(B,D)$.

<u>Thm</u> (GLS) For B Dynkin, $M \longmapsto X_M$ induces a bijection

$$\{\text{rigid ind. l.f. modules}\} \xleftrightarrow{\sim} \left\{\begin{array}{l}\text{non-initial cluster} \\ \text{variables in } \mathcal{A}(B)\end{array}\right\}.$$

<u>Rmk</u>. The proof relies on realizing $\mathcal{U}(\mathfrak{n})$ via $\operatorname{Mod}_{l.f.} H(B,D)$ (GLS) and a cluster structure on $\mathbb{C}[N]$ (Yang–Zelevinsky)

CC functions under reflection for general B

Reflection functors (GLS): when k is a sink (source)

$$F_k^{\pm} : \operatorname{mod} H(B,D) \longrightarrow \operatorname{mod} H(\mu_k B, D)$$

generalizing BGP reflections.

e.g.

$$\begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1^2=0"] \arrow[r] & 2 \end{tikzcd} \xrightarrow{\quad F_2^+ \quad} \begin{tikzcd} 1 \arrow[loop above, "\varepsilon_1^2=0"] & 2 \arrow[l] \end{tikzcd}$$

$$\begin{smallmatrix} 1 \\ 1 \\ 2 \end{smallmatrix} \longmapsto \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix}$$

$$\begin{smallmatrix} & 1 & \\ 2 & 1 & \\ & & 2 \end{smallmatrix} \longmapsto \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$$

Prop. Let k be a sink in $H(B,D)$ and $M \in \mathrm{mod}_{l.f.}\, H(B,D)$ s.t. $F_k^- F_k^+ (M) \cong M$. Then we have

$$X_M(x_1, \cdots, x_n) = X_{F_k^+ M}(x_1', \cdots, x_n')$$

where $x_i' = x_i \quad i \neq k$,

$$x_k' = x_k^{-1} \left(\prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+} \right).$$

In Dynkin cases, every cluster variable can be obtained by sink/source mutations from initial cluster variables

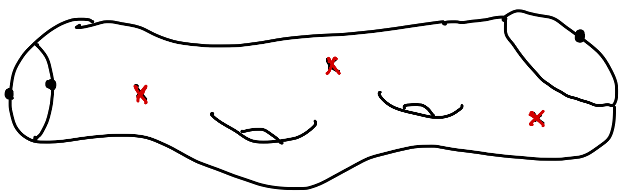
(every root can be obtained by simple reflections from simple roots)

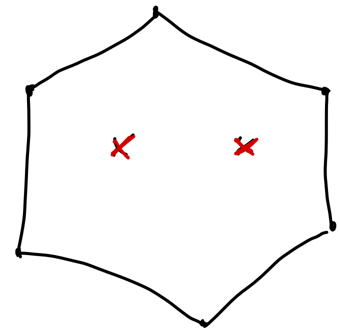
$\Downarrow$

A new proof of GLS' thm.

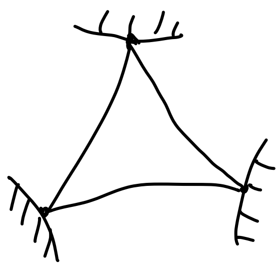
Beyond acyclic clusters (joint with D. Labardini-Fragoso)

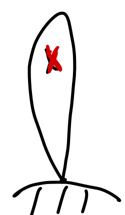
Surfaces with orbifolds



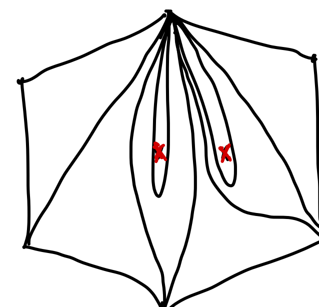


Triangulation.

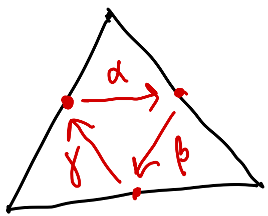


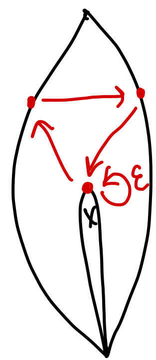


e.g.



Quivers / gentle algebras





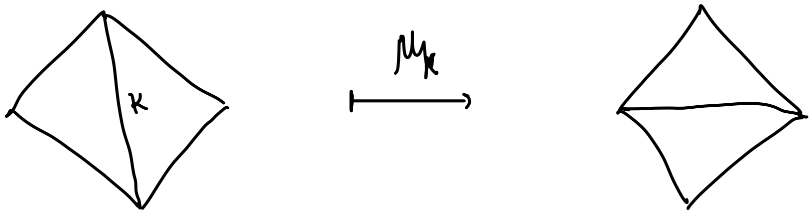
relations : $\beta\alpha$, $\gamma\beta$, $\alpha\gamma$.

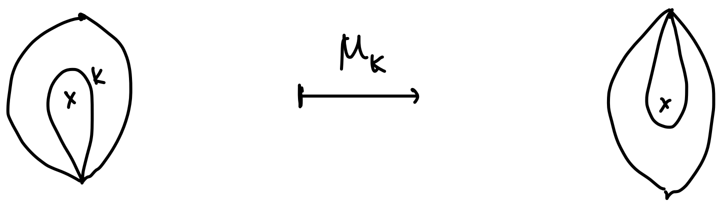
$\varepsilon^2 = 0$

Def. $H := \mathbb{C}\tilde{Q}/I$ gentle and jacobian.

$B = (\text{adjacency matrix of } Q) \cdot \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$ skew-symmetrizable.

Flips and mutations





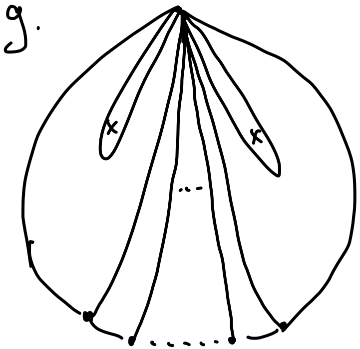
$$B \longmapsto \mu_k B$$

Thm (Labardini – M, upcoming)

$$\left\{ \tau\text{-rigid ind. } H\text{-modules (l.f.)} \right\} /\sim \quad \xleftrightarrow{\sim} \quad \left\{ \text{non-initial cluster variables of } \mathcal{A}(B) \right\}$$

$$M \longmapsto X_M := x^{g(M)} \cdot F_M(\hat{y}_1, \dots, \hat{y}_n).$$

E.g.



$\varepsilon_1^2=0$ $\qquad\qquad$ $\varepsilon_{n+1}^2=0$

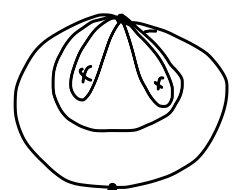
$$1 \to \cdots\cdots \to n+1$$

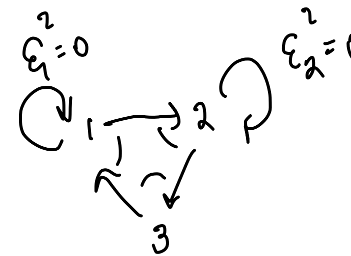
Affine type: $\tilde{C}_n$

- Proves a conjecture of GLS in $\tilde{C}_n$ that X_M is cluster variable for M τ-rigid ind.

- Extend to any initial cluster.

For example





Rmk. Since H is jacobian, one can apply DT-theory.

This recovers Chekhov - Shapiro's generalized cluster structures.

See arxiv: 2203.11563 joint with D. Labardini-Fragoso.

For l.f. ones, we are unable to apply DT-theory. have to prove recursions analogous to DWZ.

There are still lot to be discovered about "Cluster Characters" for arbitrary <u>skew-symmetrizable</u> cluster algebras.

Thank You !