TF equivalence classes constructed from canonical decompositions

Sota Asai (Osaka University)

joint work with Osamu Iyama (the University of Tokyo)

2022/04/21, fdseminar

Motivation

Let A be a fin. dim. K-algebra over a field K.

- $K_0(\text{proj}\, A)_{\mathbb{R}} := K_0(\text{proj}\, A) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.
- Each $\theta \in K_0(\text{proj}\, A)_{\mathbb{R}}$ gives an $\mathbb{R}$-linear form

$$\theta \colon K_0(\text{mod}\, A)_{\mathbb{R}} \to \mathbb{R}$$

via the Euler form $K_0(\text{proj}\, A)_{\mathbb{R}} \times K_0(\text{mod}\, A)_{\mathbb{R}} \to \mathbb{R}$.

By using this duality, the following notions were introduced:

- θ-semistable modules $M \in \text{mod}\, A$ by [King]
  $\to$ Wall-chamber structures on $K_0(\text{proj}\, A)_{\mathbb{R}}$ by [BST, Bridgeland].
- Two numerical torsion pairs in $\text{mod}\, A$ for each θ by [BKT]
  $\to$ TF equivalence on $K_0(\text{proj}\, A)_{\mathbb{R}}$ by [A].

These two are strongly related to each other.
To study them, silting theory is useful.

TF equiv. classes by presilting complexes

Let $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ be 2-term presilting with U_i: indec.
We set the presilting cone of U by

$$C^+(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i] \subset K_0(\operatorname{proj} A)_{\mathbb{R}}.$$

Proposition [Brüstle-Smith-Treffinger, Yurikusa, (A)]

For each $U \in$ 2-psilt A, $C^+(U)$ is a TF equivalence class.

However, presilting cones do not give all TF equivalence classes
if A is not τ-tilting finite [Zimmermann-Zvonareva].

Today's theme

To obtain more TF equivalence classes,
we use canonical decompositions by [Derksen-Fei].

Canonical decompositions

We use the presentation space for each $\theta \in K_0(\mathrm{proj}\, A)$:

$$\mathrm{Hom}(\theta) := \mathrm{Hom}_A(P_+, P_-),$$

where $\theta = [P_+] - [P_-]$ and $\mathrm{add}\, P_+ \cap \mathrm{add}\, P_- = \{0\}$.
Each $f \in \mathrm{Hom}(\theta)$ defines a 2-term complex

$$P_f := (P_- \xrightarrow{f} P_+) \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A).$$

[Derksen-Fei] defined direct sums in $K_0(\mathrm{proj}\, A)$:

$$\bigoplus_{i=1}^{m} \theta_i :\Longleftrightarrow \begin{bmatrix} \text{For general } f \in \mathrm{Hom}(\sum_{i=1}^{m} \theta_i), \\ \exists f_i \in \mathrm{Hom}(\theta_i),\ P_f \cong \bigoplus_{i=1}^{m} P_{f_i} \end{bmatrix}.$$

This is called a canonical decomposition if each θ_i is indecomposable.

Theorem [DF, Plamondon]

Any $\theta \in K_0(\mathrm{proj}\, A)$ admits a unique canon. decomp. $\bigoplus_{i=1}^{m} \theta_i$.

Our results

We introduced E-tame algebras in our study:

$$A\text{: E-tame} :\Longleftrightarrow \forall\theta \in K_0(\operatorname{proj} A),\ \theta \oplus \theta.$$

All representation-tame algebras are E-tame [GLFS].

Today's main theorem [AI]

Assume that A is hereditary or E-tame.
Let $\theta = \bigoplus_{i=1}^{m} \theta_i$ be a canon. decomp. in $K_0(\operatorname{proj} A)$.
Then, $C^+(\theta) := \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$ is a TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

If $\theta_i \neq \theta_j$ for any $i \neq j$ in above, then $\theta_1, \ldots, \theta_m$ are lin. independent.

Setting

Let A be a fin. dim. algebra over an alg. closed field K.

- proj A: the category of fin. gen. projective A-modules.
- $P_1, P_2, \dots, P_n$: the non-iso. indec. proj. modules.
- $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$: the homotopy cat. of bounded complexes over proj A.
- mod A: the category of fin. gen. A-modules.
- $S_1, S_2, \dots, S_n$: the non-iso. simple modules
  (we may assume there exists a surj. $P_i \longrightarrow S_i$).
- $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\, A)$: the derived cat. of bounded complexes over mod A.
- $K_0(\mathcal{C})$: the Grothendieck group of $\mathcal{C}$.
- $K_0(\mathcal{C})_{\mathbb{R}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.

The Euler form

$K_0(\text{proj}\, A)$ and $K_0(\text{mod}\, A)$ are free abelian groups.

Proposition (see [Happel])

(1) $K_0(\text{proj}\, A) = K_0(\mathsf{K}^{\mathrm{b}}(\text{proj}\, A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$.

(2) $K_0(\text{mod}\, A) = K_0(\mathsf{D}^{\mathrm{b}}(\text{mod}\, A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$.

(3) $\langle [P_i], [S_j] \rangle = \delta_{i,j}$, where

$$\langle \cdot, \cdot \rangle \colon K_0(\text{proj}\, A) \times K_0(\text{mod}\, A) \to \mathbb{Z}$$

is the Euler form.

These are naturally extended to the real Grothendieck groups.
Via the Euler form, each $\theta \in K_0(\text{proj}\, A)_{\mathbb{R}}$ induces the $\mathbb{R}$-linear form

$$\theta := \langle \theta, \cdot \rangle \colon K_0(\text{mod}\, A)_{\mathbb{R}} \to \mathbb{R}.$$

Wall-chamber structures

Definition [King]

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.

(1) $M \in \operatorname{mod} A$: θ-semistable $:\Longleftrightarrow$
$\theta(M) = 0$ and $\theta(N) \geq 0$ for any quotient N of M.

(2) $\mathcal{W}_\theta := \{\text{all } \theta\text{-semistable modules}\} \subset \operatorname{mod} A$.

Definition [Brüstle-Smith-Treffinger, Bridgeland]

(1) For $M \in \operatorname{mod} A \setminus \{0\}$, set $\Theta_M := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid M \in \mathcal{W}_\theta\}$.

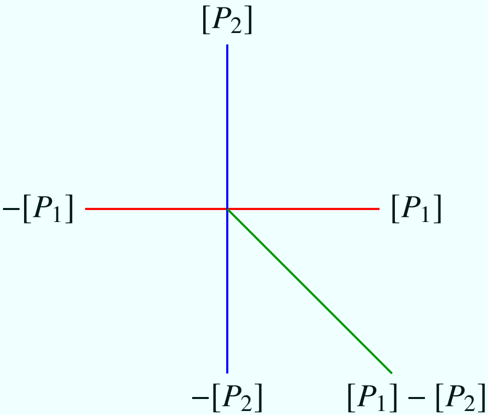
(2) We consider the wall-chamber structure on $K_0(\operatorname{proj} A)_{\mathbb{R}}$
whose walls are Θ_M for all $M \in \operatorname{mod} A \setminus \{0\}$.

Remark

To get the wall-chamber structure,
it suffices to consider indec. modules.

Example of walls

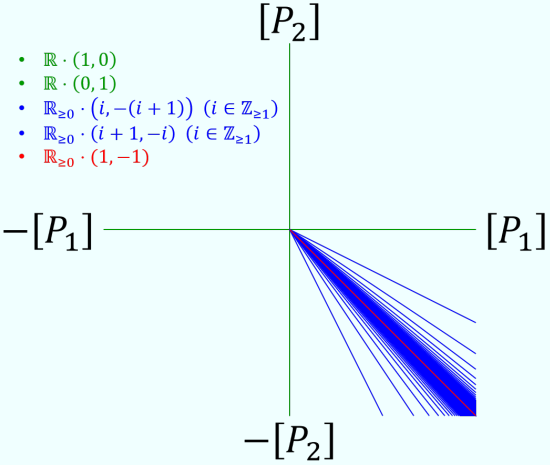
Let $A = K(1 \to 2)$, then the indec. modules are S_2, P_1, S_1.



There are 5 chambers.

Example of walls

Let $A = K(\ 1 \rightrightarrows 2\ )$.



There are infinitely many chambers.

TF equivalence

Definition [Baumann-Kamnitzer-Tingley]

Let $\theta \in K_0(\text{proj } A)_{\mathbb{R}}$.
We define numerical torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ in $\text{mod } A$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \text{mod } A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \text{mod } A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \text{mod } A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \text{mod } A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

Definition

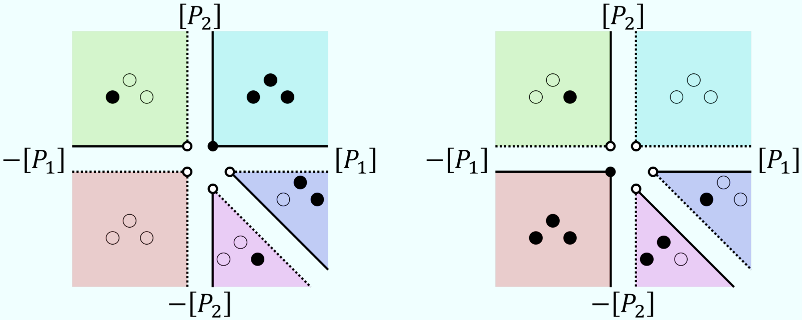
$\theta, \theta' \in K_0(\text{proj } A)_{\mathbb{R}}$ are TF equivalent $:\Longleftrightarrow$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'}), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'}).$$

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{smallmatrix} & P_1 & \\ S_2 & & S_1 \end{smallmatrix}$ are the indec. A-modules.

Then, $\overline{\mathcal{T}}_\theta$ and $\overline{\mathcal{F}}_\theta$ are given as follows.

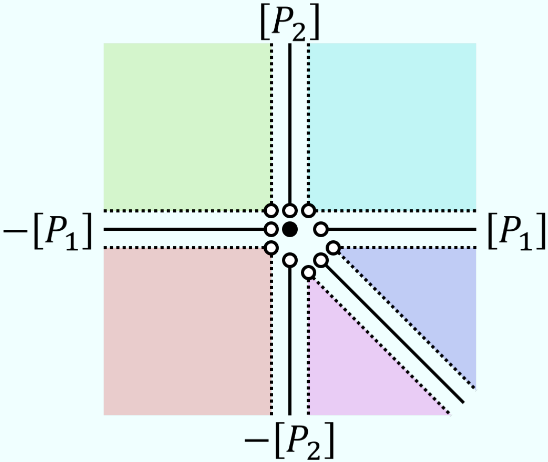


(●: belong, ○: not belong)

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{matrix} & P_1 & \\ S_2 & & S_1 \end{matrix}$ are the indec. A-modules.

There are exactly 11 TF equivalence classes.



Walls and TF equiv. classes

Proposition [A]

Let $\theta \neq \theta' \in K_0(\text{proj}\, A)_{\mathbb{R}}$, then TFAE.

(a) θ and θ' are TF equivalent.

(b) $\mathcal{W}_{\theta''}$ is constant for $\theta'' \in [\theta, \theta']$.

(c) $\nexists S \in \text{brick}\, A$, $[\theta, \theta'] \cap \Theta_S$ is one point.

Example

If $A = K(\ 1 \rightrightarrows 2\ )$, then the TF equivalence classes are

- $\{0\}$,
- $\mathbb{R}_{>0}(i, -(i+1)), \mathbb{R}_{>0}(i+1, -i)$,
- $\mathbb{R}_{>0}(i, -(i+1)) + \mathbb{R}_{>0}(i+1, -(i+2)), \mathbb{R}_{>0}(i+1, -i) + \mathbb{R}_{>0}(i+2, -(i+1))$,
- $\mathbb{R}_{>0}(1, -1)$

where we consider all $i \in \mathbb{Z}_{\geq 0}$.

Presilting complexes

Definition [Keller-Vossieck]

Let $U = (U^{-1} \to U^0) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ be a 2-term complex.

(1) U: presilting $:\Longleftrightarrow$ $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(U, U[1]) = 0$.

(2) U: silting $:\Longleftrightarrow$ U: presilting, $\operatorname{thick}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)} U = \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

2-psilt $A := \{$basic 2-term presilting complexes$\}/\cong$.
2-silt $A := \{$basic 2-term silting complexes$\}/\cong$.

Proposition [(1) Aihara, (2) Adachi-Iyama-Reiten]

(1) $\forall U \in$ 2-psilt A, $\exists T \in$ 2-silt A s.t.
U is a direct summand of T.

(2) $U \in$ 2-silt $A \iff U \in$ 2-psilt A, $|U| = n$
($|U|$ means the number of non-iso. indec. direct summands of U).

Presilting and func. fin. torsion pairs

For each $U \in$ 2-psilt A, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^{\perp}H^{-1}(\nu U), \mathrm{Sub}\, H^{-1}(\nu U)),$$
$$(\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathrm{Fac}\, H^0(U), H^0(U)^{\perp}).$$

Then, $\mathcal{T}_U \subset \overline{\mathcal{T}}_U$ and $\mathcal{F}_U \subset \overline{\mathcal{F}}_U$.

Theorem [Smalø, Auslander-Smalø, AIR]

Let $U \in$ 2-psilt A.

(1) $(\overline{\mathcal{T}}_U, \mathcal{F}_U)$, $(\mathcal{T}_U, \overline{\mathcal{F}}_U)$ are func. fin. torsion pairs.

(2) All func. fin. torsion(-free) classes are obtained in this way.

Presilting cones

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i: indec.

Proposition [Aihara-Iyama]

$[U_1], \ldots, [U_m] \in K_0(\text{proj } A)$ are linearly independent.
If $U \in$ 2-silt A, they are a $\mathbb{Z}$-basis of $K_0(\text{proj } A)$.

Definition

We define the presilting cone $C^+(U)$ in $K_0(\text{proj } A)_{\mathbb{R}}$ by

$$C^+(U) := \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i].$$

Proposition [Demonet-Iyama-Jasso]

If $U \neq U' \in$ 2-psilt A, then $C^+(U) \cap C^+(U') = \emptyset$.

Presilting cones are TF equiv. classes

Theorem (⇒): [Yurikusa, Brüstle-Smith-Treffinger], (⇐): [A]

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i indec.
Then, $C^+(U)$ is a TF equiv. class such that

$$\eta \in C^+(U) \iff \overline{\mathcal{T}}_\eta = \overline{\mathcal{T}}_U,\ \overline{\mathcal{F}}_\eta = \overline{\mathcal{F}}_U.$$

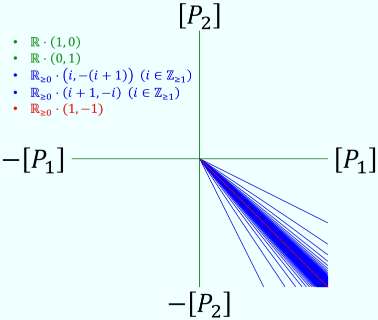
Theorem [A]

The following sets coincide.

- The set of chambers in the wall-chamber structures.
- The set of TF equiv. classes whose interiors are nonempty.
- $\{C^+(T) \mid T \in$ 2-silt $A\}$.

Example of presilting and TF equiv. classes

Let $A = K(\ 1 \rightrightarrows 2\ )$.



The TF equivalence classes in $K_0(\text{proj}\, A)_{\mathbb{R}}$ are

- $C^+(U)$ for all $U \in$ 2-psilt A,
- $\mathbb{R}_{>0}(1, -1)$ (this does not come from 2-psilt A).

Presentation spaces

Definition [Derksen-Fei]

Let $\theta \in K_0(\text{proj}\, A)$.

(1) Take $P_+, P_- \in \text{proj}\, A$ (unique up to iso.) such that $\theta = [P_+] - [P_-]$ and $\text{add}\, P_+ \cap \text{add}\, P_- = \{0\}$.

(2) $\text{Hom}(\theta) := \text{Hom}_A(P_-, P_+)$: the presentation space of θ.

(3) For each $f \in \text{Hom}(\theta)$, set $P_f := (P_- \xrightarrow{f} P_+) \in \mathsf{K}^{\mathrm{b}}(\text{proj}\, A)$ (the terms except -1st and 0th ones vanish).

$\text{Hom}(\theta)$ is an irreducible algebraic variety.

Convention

"Any general $f \in \text{Hom}(\theta)$ satisfies (P)" means
"there exists $X \subset \text{Hom}(\theta)$: nonempty and open (thus dense) such that any $f \in X$ satisfies (P)".

Direct sums in $K_0(\text{proj } A)$

Definition [DF]

We say a direct sum $\bigoplus_{i=1}^m \theta_i$ holds in $K_0(\text{proj } A)$ if

$$\text{for general } f \in \text{Hom}\left(\sum_{i=1}^m \theta_i\right), \exists f_i \in \text{Hom}(\theta_i), P_f \cong \bigoplus_{i=1}^m P_{f_i}.$$

In this case, we also write $\sum_{i=1}^m \theta_i = \bigoplus_{i=1}^m \theta_i$.

This condition can be checked pairwisely.

Proposition [DF]

$\bigoplus_{i=1}^m \theta_i \iff \forall i \neq j, \ \exists (f, g) \in \text{Hom}(\theta_i) \times \text{Hom}(\theta_j),$

$$\text{Hom}(P_f, P_g[1]) = 0, \quad \text{Hom}(P_g, P_f[1]) = 0.$$

Canonical decompositions

Definition

θ: indecomposable in $K_0(\operatorname{proj} A)$:$\Longleftrightarrow$
for any general $f \in \operatorname{Hom}(\theta)$, $P_f \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is indec.

Theorem [DF, Plamondon]

Any $\theta \in K_0(\operatorname{proj} A)$ admits a decomposition unique up to reordering

$$\theta = \bigoplus_{i=1}^{m} \theta_i \quad (\theta_i\text{: indecomposable}).$$

We call it the canonical decomposition of θ.

Our results 1

Theorem 1 [AI] (with Demonet)

Let $\bigoplus_{i=1}^{m} \theta_i$ in $K_0(\text{proj}\, A)$. Then,

$$\eta \in \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i \Longrightarrow \overline{\mathcal{T}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{T}}_{\theta_i},\ \overline{\mathcal{F}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{F}}_{\theta_i}.$$

Thus, for any i, $\mathcal{T}_{\theta_i} \subset \mathcal{T}_\eta \subset \overline{\mathcal{T}}_\eta \subset \overline{\mathcal{T}}_{\theta_i}$, $\mathcal{F}_{\theta_i} \subset \mathcal{F}_\eta \subset \overline{\mathcal{F}}_\eta \subset \overline{\mathcal{F}}_{\theta_i}$.

We can recover the following sign-coherence.

Proposition [Plamondon]

Let $\theta \oplus \theta'$ in $K_0(\text{proj}\, A)$, $\theta = \sum_{i=1}^{n} a_i[P_i]$ and $\theta' = \sum_{i=1}^{n} a'_i[P_i]$.
Then, $a_i a'_i \geq 0$ for all i.

$\because$ If $a_i > 0$ and $a'_i < 0$, then $S_i \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \subset \mathcal{T}_{\theta+\theta'} \cap \mathcal{F}_{\theta+\theta'} = \{0\}$.

Our results 2

By Theorem 1, if $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$, then

$$C^+(\theta) := \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$$

is contained in some TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.
Is $C^+(\theta)$ really a TF equiv. class?

Theorem 2 [AI]

Assume that

- A is a hereditary algebra; or
- A is E-tame, i.e. $\theta \oplus \theta$ holds for any $\theta \in K_0(\operatorname{proj} A)$.

If $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$,
then $C^+(\theta)$ is a TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

E-tame algebras

Though it is not easy to check the E-tameness, we have the following.

Theorem [Geiss-Labardini-Fragoso-Schröer, (Plamondon-Yurikusa)]

Let A be representation-finite or tame.
Then, A is E-tame.

Why did we assume E-tameness?
Because our proof of Theorem 2 uses the following result.

Theorem [Fei]

If $\theta \in K_0(\text{proj } A)$ and $M \in \text{mod } A$, then TFAE.

(a) $M \in \overline{\mathcal{F}}_\theta$.

(b) $\exists l \in \mathbb{Z}_{\geq 1}, \exists f \in \text{Hom}(l\theta), \text{Hom}_A(\text{Coker } f, M) = 0$.

Moreover, we may let $l = 1$ if $\theta \oplus \theta$.

Example of Theorem 2

Let Q be an extended Dynkin quiver, and $A := KQ$.

- Consider an indec. module $M \in \operatorname{mod} A$ in a regular homog. tube.
- Take the min. proj. resol. $P_1^M \to P_0^M \to M \to 0$, and set $\eta := [P_0^M] - [P_1^M]$.
- $E := \{U \in \text{2-psilt}\, A \mid [U] \oplus \eta\}$.
    - $[U] \oplus \eta \iff [U] \in \Theta_M \iff H^0(U), H^{-1}(\nu U)$ are regular.

Proposition

Under the setting above, the TF equiv. classes in $K_0(\operatorname{proj} A)_{\mathbb{R}}$ are

- $C^+(U)$ for all $U \in \text{2-psilt}\, A$ and
- $C^+([U] \oplus \eta) = C^+(U) + \mathbb{R}_{>0}\eta$ for all $U \in E$.

In particular, all TF equiv. classes come from canon. decomp.

Final remark

In general, even if A is E-tame,
Theorem 2 does not necessarily give all TF equiv. classes.

- We cannot obtain any TF equiv. class $E \subset K_0(\text{proj } A)_{\mathbb{R}}$ such that $E \cap K_0(\text{proj } A) = \emptyset$ from Theorem 2.
- The following gentle algebra admits a TF equiv. class $\mathbb{R}_{>0}(1-t, -1+2t, -t)$ for each $t \in [0,1] \setminus \mathbb{Q}$:

$$A = K\left(1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3 \right)/\langle \alpha\delta, \beta\gamma \rangle.$$

Thank you for your attention.

s-asai@ist.osaka-u.ac.jp