Weight structures and geometric representation theory

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HANDOUT

jt with J. Eberhardt

① Convolution

$X_1, X_2, \ldots$ smooth varieties $/ k = \bar{k}$

$\mu_1 \downarrow \; \downarrow \mu_2 \; \ldots$ $\quad \mu_i$ proper

W not necessarily smooth

$$\begin{tikzcd}
X_j \times_W X_k & & & \\
\times & X_i \times_W X_j \times X_j \times_W X_k \arrow[ul, "pr"'] \arrow[dl, "pr"] & X_i \times_W X_j \times_W X_k \arrow[l, "1 \times \Delta \times 1"'] \arrow[r, "pr"] & X_i \times_W X_k \\
X_i \times_W X_j & & &
\end{tikzcd}$$

$$(\alpha, \beta) \longmapsto \alpha * \beta := pr_* \circ \Delta^! ((\alpha, \beta))$$

$$\begin{matrix} Ch(X_j \times_W X_k) \\ \times \\ Ch(X_i \times_W X_j) \end{matrix} \xrightarrow{\text{convolution}} Ch(X_i \times_W X_j)$$

Chow groups of cocycles / rational equivalence (work with $\mathbb{Q}$ coefficients)

Examples

1) $$\begin{array}{c}\tilde{\mathcal{N}} \\ \mu \big\downarrow \\ \mathcal{N}\end{array}$$ Springer resolution

$\mathcal{N}$ = nilpotent elements in ss. Lie algebra $\mathfrak{g}$

$$E := Ch^0(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}) = \mathbb{Q}[\text{Weyl group}]$$

2) $Q = (Q_0, Q_1)$ quiver

$$\begin{array}{c} Q(d) := \{\text{flagged representations of dim vector } d \text{ and flag type } \underline{d}\} \\ \text{forget flag} = \mu \big\downarrow \\ Rep_d \end{array}$$

$\underline{d}$: vector composition of d

$G := \prod_{i \in Q_0} Gl_{d_i}$ - equivariant

$$\boxed{E := \bigoplus_{\underline{d}, \underline{d}'} Ch^G(Q(\underline{d}) \times_{Rep_d} Q(\underline{d}'))}$$

$\underline{d}, \underline{d}'$: vector compositions of d

Motivic KLR-algebra ($\rightsquigarrow$ Khovanov-Lauda Rouquier Varagnolo-Vasserot)

Motivic Quiver Schur algebra (S.-Webster)

3) $X = G/B$ flag variety

$\cup$

$\overline{X}_w$ Schubert variety (w ∈ Weyl group)

$\cup$

X_w Schubert cell

$$\begin{array}{ll} BS(w) & \\ \downarrow \mu_w & = \text{Bott-Samelson resolution of } X_w \\ X & \text{T-equivariant} \end{array}$$

$\rightsquigarrow E := \bigoplus_{w,w'} Ch(BS(w) \times_{G/B} BS(w'))$

endomorphism algebra of certain Soergel bimodules

4) (Graded) Hecke algebras (Lusztig)

Theorem (Eberhardt–S.) [Formality]

Setup: $\tilde{\mathcal{N}}_i$ $\quad i \in I$ smooth variety $/\overline{\mathbb{F}}_p$

$\quad \downarrow \mu_i$ proper, G-equiv

$\mathcal{N}$ (not necess. smooth)

G affine alg grp

$$E := \bigoplus_{i,j \in I} \mathrm{Ch}\left(\tilde{\mathcal{N}}_i \times_{\mathcal{N}} \tilde{\mathcal{N}}_j\right) \quad \text{algebra}$$

Assume

(PT) $\quad M(\mu^{-1}(x)) \in \langle \mathbb{Q}(n)[2n] \rangle_{\oplus, \subset_{\oplus}}$

(PO) $\quad \mu_i(\tilde{\mathcal{N}}_i) \subseteq \mathcal{N}$ has finitely many G-orbits

Then

weight complex functor

$$\underbrace{D^{\mathbb{Z}}_{\mathrm{perf}}(E)}_{\text{perfect derived category of } \mathbb{Z}\text{-graded } E\text{-modules}} \xleftarrow{\sim} \underbrace{DM^{\mathrm{Spr}}_G(\mathcal{N})}_{\text{Springer motives}} \subseteq \underbrace{DM_G(\mathcal{N})}_{\text{derived cat. of } G\text{-equiv. motivic sheaves on } \mathcal{N}}$$

Theorem (ES): Assumptions hold in above examples assuming type $\tilde{A}$ A D E in 2)

(uses results of

De Concini – Lusztig

Cerulli-Irelli-Esposito-Franzen-Reineke

Maksimau)

Weight structures

Definition A.2. [Bon10, Definition 1.1.1] *Let $\mathcal{C}$ be a triangulated category. A* weight structure $\mathbf{w}$ *on $\mathcal{C}$ is a pair $\mathbf{w} = (\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0})$ of full subcategories of $\mathcal{C}$, which are closed under direct summands, such that with $\mathcal{C}^{w\leq n} := \mathcal{C}^{w\leq 0}[-n]$ and $\mathcal{C}^{w\geq n} := \mathcal{C}^{w\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{w\leq 0} \subseteq \mathcal{C}^{w\leq 1}$ and $\mathcal{C}^{w\geq 1} \subseteq \mathcal{C}^{w\geq 0}$;
(2) for all $X \in \mathcal{C}^{w\geq 0}$ and $Y \in \mathcal{C}^{w\leq -1}$, we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$;
(3) for any $X \in \mathcal{C}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \xrightarrow{+1}$$

with $A \in \mathcal{C}^{w\geq 1}$ and $B \in \mathcal{C}^{w\leq 0}$.
The full subcategory $\mathcal{C}^{w=0} = \mathcal{C}^{w\leq 0} \cap \mathcal{C}^{w\geq 0}$ is called the heart *of the weight struture.*

Definition A.1. [BBD82, Definition 1.3.1] *Let $\mathcal{C}$ be a triangulated category. A* t-structure t *on $\mathcal{C}$ is a pair $t = (\mathcal{C}^{t\leq 0}, \mathcal{C}^{t\geq 0})$ of full subcategories of $\mathcal{C}$ such that with $\mathcal{C}^{t\leq n} := \mathcal{C}^{t\leq 0}[-n]$ and $\mathcal{C}^{t\geq n} := \mathcal{C}^{t\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{t\leq 0} \subseteq \mathcal{C}^{t\leq 1}$ and $\mathcal{C}^{t\geq 1} \subseteq \mathcal{C}^{t\geq 0}$;
(2) for all $X \in \mathcal{C}^{t\leq 0}$ and $Y \in \mathcal{C}^{t\geq 1}$, we have $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$;
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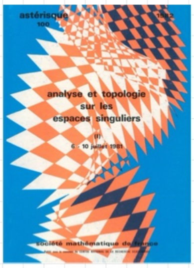
$$A \longrightarrow X \longrightarrow B \xrightarrow{+1}$$

with $A \in \mathcal{C}^{t\leq 0}$ and $B \in \mathcal{C}^{t\geq 1}$.
The full subcategory $\mathcal{C}^{t=0} = \mathcal{C}^{t\leq 0} \cap \mathcal{C}^{t\geq 0}$ is called the heart of the t-struture.

Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)

M.V. Bondarko



also require all categories to be idempotent complete

General construction (Bondarko)

$\mathcal{J} \subseteq \mathcal{C}$ triang. cat, idempotent complete

Collection of objects

Assume: • $\mathcal{J}$ negative i.e $\mathrm{Hom}(X, Y[n]) = 0 \quad n > 0 \quad \forall X, Y \in \mathcal{J}$

• $\langle \mathcal{J} \rangle_\Delta = \mathcal{C}$

$\Rightarrow$ weight structure with $\heartsuit = \mathcal{C}^{\omega=0} = \langle \mathcal{J} \rangle_{\cong, \oplus, \ominus}$

Beilinson's realisation functor:

$$\boxed{D^b(\mathcal{C}^{t=0}) \longrightarrow \mathcal{C}}$$

t-structure

Bondarko's weight complex functor:

$$\boxed{wt: \mathcal{C} \longrightarrow K^b(\mathcal{C}^{w=0})}$$

$\heartsuit$

⊛ Class: bounded weight structure, $\mathcal{C} = h\mathcal{C}_\infty$)

Prop: Assume ⊛. Then

$$\boxed{\begin{array}{l} wt \text{ is an equivalence} \\ \Leftrightarrow \mathcal{C}^{w=0} \text{ is tilting} \\ \Leftrightarrow \operatorname{Hom}(M, N[i]) = 0 \quad \forall i \neq 0 \quad \forall M, N \in \mathcal{C}^{w=0}. \end{array}}$$

(Chow) motives

Define category of correspondences (over $\mathcal{N}$)

$$\mathrm{Corr}_G(\mathcal{N}) = \begin{cases} \text{objects: } M(X/\mathcal{N}) \text{ for } X \xrightarrow{\text{proper}} \mathcal{N}, \ X \text{ smooth} \quad G\text{-equiv.} \\ \text{morphisms: } \operatorname{Hom}_{\mathrm{Corr}_G(\mathcal{N})}(M(X/\mathcal{N}), M(Y/\mathcal{N})) = CH^G(X \times_{\mathcal{N}} Y) \end{cases}$$

additive category

Can take Karoubian closure $\mathrm{Kar}(\mathrm{Corr}_G(\mathcal{N}))$

e.g. $\mathcal{M}(\mathbb{P}^1/\mathcal{N}) = \mathbb{Q} \oplus \underbrace{\mathbb{L}}_{\text{Lefschetz motive}}$

$\mathrm{Chow}_G(\mathcal{N}) := \mathrm{Kar}(\mathrm{Corr}_G(\mathcal{N}))[\mathbb{L}^{\otimes -1}]$

$\boxed{G\text{-equivariant Chow motives}}$

Bondarko: G-equiv. Chow motives form ♡ of weight structure on triangulated cat. $DM_G(\mathcal{N})$ = derived cat of G-equivariant geometric motives over $\mathcal{N}$ (= motivic sheaves on $\mathcal{N}$)

Main point behind formality theorem:

Springer motives are a certain tilting family inside $\mathcal{DM}_{G}(\mathcal{N})$