Stable Invariance of Structures on Hochschild Cohomology

Joint with

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Part 1 Stable invariance of the "p power structure" on $HH^*(A)$

Part 2 Maximal tori in $HH^1(A)$ and the fundamental group

A is finite dim algebra over a field $k = \bar{k}$ (not needed)

$A^{ev} = A^{op} \otimes A$ and $HH^*(A) = \mathrm{Ext}^*_{A^{ev}}(A, A)$

is the Hochschild cohomology of A.

Fact $HH^1(A) \cong \dfrac{\mathrm{Der}_k(A,A)}{\left(\begin{matrix}\text{inner derivations} \\ [a,-]\end{matrix}\right)}$ is a finite dimensional Lie algebra over k

bracket: $[\alpha,\beta] = \alpha\beta - \beta\alpha$

Recent interest in the structure of $HH^1(A)$ (EG RyD, Linckelmann, Schroll, Solotar, Chaparro, Benson, Kessar, Eisele, Raedschelders ...)

If $\operatorname{char} k = p > 0$ then $HH^1(A)$ is a <u>restricted</u> Lie algebra

<u>ie</u> there is a p-power operation $HH^1(A) \to HH^1(A) : \alpha \mapsto \alpha^p$

check:

$$\alpha^p(xy) = \sum \binom{p}{i} \alpha^i(x)\alpha^{p-i}(y) = \alpha^p(x)y + x\alpha^p(y) \text{ is a derivation}$$

(and this respects inner derivations)

In fact all of $HH^{>0}(A)$ is a graded restricted Lie algebra.

But focus on $HH^1(A)$:

Example $A = \frac{k[x]}{x^p}$ $\quad \partial_x(x^p) = px^{p-1} = 0$ $\quad [x^i\partial_x, x^j\partial_x] = (j-i)x^{i+j-1}\partial_x$

$\Rightarrow HH^1(A) = \mathrm{span}\{\partial_x, x\partial_x, \ldots x^{p-1}\partial_x\}$ $\quad (x^i\partial_x)^p = \begin{cases} 0 & i \neq 1 \\ x\partial_x & i = 1 \end{cases}$

This is the Jacobson–Witt Lie algebra (see RyD – Linckelmann '18)

Motivation: Need the p power operation to do Lie theory in positive characteristic. eg $T \subseteq L$ is called a torus if $[T,T]=0$ and T is generated by elements with $x^{p}=x$

if k perfect

EG $(x\partial_x)$ is the only $\neq 0$ torus in the JW Lie algebra above.

rank = max dim of torus = 1 in this case.

use this in second half.

Assume A is <u>self injective</u>: then the stable module category $\underline{\mathrm{mod}}\, A$ is triangulated

cat of fg A-modules with $\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N) / \begin{pmatrix}\text{maps which factor}\\ \text{through a projective}\end{pmatrix}$

<u>Defn</u> (Broué 94) A <u>stable equivalence of Morita type</u> $A \sim_{\mathrm{SEMT}} B$

is a pair of bimodules ${}_A M_B$ ${}_B N_A$ each projective on either side

$M \otimes_B N \simeq A$ in $\underline{\mathrm{mod}}\, A^{ev}$ & $N \otimes_A M \simeq B$ in $\underline{\mathrm{mod}}\, B^{ev}$.

This induces an equivalence of Δ'd cats

$$\underline{\mathrm{mod}}\ A \underset{-\otimes_B N}{\overset{-\otimes_A M}{\rightleftarrows}} \underline{\mathrm{mod}}\ B$$

problem: does every equivalence come from a SEMT?

these are more general than derived equivalences

Example $A_4 \subseteq A_5$, char $k=2$

induces a SEMT $kA_4 \sim kA_5$ using $M = kA_5$

"because A_4 A_5 have the same sylow 2 subgroups" (See Linckelmann's book)

Big problem classify algebras (or groups) up to SEMT.

$\Rightarrow$ want invariants (eg because Auslander Reiten conjecture)

Hochschild cohomology is a derived Morita invariant

but $HH^0(A) = Z(A)$ not stably invariant:

eg $Z(kA_4) \not\cong Z(kA_5)$

Theorem (Xi '08, König, Liu, Zhou '12)

A, B fd (self inj) symmetric algebras and $A \sim_{SEMT} B$, there is an isom

$$tr_M : HH^{>0}(A) \xrightarrow{\cong} HH^{>0}(B)$$ respecting the cup product and Gerstenhaber bracket

"transfer map, Bouc" also Linckelmann

Xi

KLZ

- proof uses BV operator $\Delta : HH^* \to HH^{*-1}$

$$\Delta(xy) - \Delta(x)y - (-1)^{|x|} x \Delta(y) = [x, y]$$

- But it is impossible to write the p-power operation in terms of Δ (RyD thesis)

(And transfer maps do not respect p-power map in general) — RyD '17

So:

Question (Linckelmann) Does the transfer map associated to a SEMT respect the p-power map?

(RyD '17) Yes for $HH^1_{int}(A) \subseteq HH^1(A)$ the "integrable derivations"

interesting but a different story

<u>Theorem</u> (- RyD '20)

If A, B are f.d. self injective algebras, $\operatorname{char} k = p$, and $A \underset{SEMT}{\sim} B$

then $tr_M : HH^{>0}(A) \xrightarrow{\cong} HH^{>0}(B)$ isom of <u>restricted</u> graded Lie algebras.

<u>So</u> the <u>restricted Lie algebra</u> $HH^1(A)$ is a <u>stable invariant</u>.

<u>Corollary</u> maximal tori in $HH^1(A)$ $\longleftrightarrow$ maximal tori in $HH^1(B)$ $\Leftarrow$ will use in part 2

__the proof__ • uses the B_∞ structure on the Hochschild cochain complex $C^*(A)$

Compare later work of Chen, Li, Wang

• the p-power operation __can__ be expressed in terms of the B_∞ structure (work of Turchin '06 also appendix of our paper)

__Note__: B_∞ algebras defined by Jones - Getzler

__But__ Gerstenhaber defined essentially the same thing in "On the defs of rings and algebras III" 68 ← __cool stuff.__

called them __composition complexes__.

Part 2 the fundamental group:

Let Q be a quiver, $I \subseteq kQ$ admissible ideal,

$A = \frac{kQ}{I}$ fd basic algebra

- kQ has a basis of paths $\{p_i\}$
- I has a basis of minimal relations $\{ r = \sum a_i p_i \}$

  no proper sub-sum of r is in I

- say $p_i \sim p_j$ if they both occur in a minimal relation.

- a <u>walk</u> is a path in $Q \cup Q^{-1}$

  up to the equivalence relation generated by

  - $x a a^{-1} y \sim xy$
  - $x p_i y \sim x p_j y$ if $p_i \sim p_j$

- $$\pi_1(Q, I) = \frac{\{\text{walks } v \to v\}}{\sim}$$ this is a <u>fg</u> <u>group</u> using concatenation.

Pick $v \in Q_0$ vertex : doesn't depend on e up to isom

This definition is due to Martinez-Villa and de la Peña.

Example

if $I = (0)$ $\quad \pi_1(Q, I) = \pi_1(|Q|)$

the topological fundamental group of the underlying graph of Q

Note: the same holds if I is monomial

Example (from Le Meur '05)

$$Q = \begin{tikzcd} & \bullet \arrow[dr, "c"] & & \\ \bullet \arrow[ur, "b"] \arrow[rr, "a"'] & & \bullet \arrow[r, "d"] & \bullet \end{tikzcd}$$

$I = (da)$ $\qquad J = (da - dcb)$

$\pi_1(Q, I) \cong \mathbb{Z}$ $\qquad \pi_1(Q, J) \cong 0$

But

$kQ/I \cong kQ/J$

So the fundamental group depends on the choice of presentation...

we want to use π_1 as a stable invariant that tells us about the shape of the quiver, but it isn't even an invariant of A. Don't worry!

As (Q, I) varies (moduli space of presentations of A) you different $\pi_1(Q, I)$ $\rightarrow$ generically zero

$\rightarrow$ special pts get <u>maximal fundamental groups</u> (Le Meur)

Theorem (Assem–de la Peña '96, de la Peña Saorín '00)

For any presentation $A = kQ/I$ there is a cannonical embedding

$$\mathrm{Hom}_k(\pi_1(Q\,I), k) \hookrightarrow HH^1(A)$$

and the image is a torus in $HH^1(A)$.

Question: which tori do you get?

(Le Meur '10) You get all the maximal tori if either

- k has char 0 and Q has no double bypasses
- A is monomial and Q has no oriented cycles and no parallel arrows

$\exists$ unique maximal π_1

π_1

Theorem (– RyD Samin 21)

For any A, every maximal torus in $HH^1(A)$ is the image of some $\pi_1(Q,I)^*$.

Proof uses Lagrange interpolation.

Get a correspondence:

$$\text{maximal fundamental groups of } A \longleftrightarrow \text{maximal tori of } HH^1(A)$$

Cor the maximal rank* of $\pi_1(Q,I)$, for any presentation $A = \frac{kQ}{I}$ is a stable invariant ← uses part 1

* **Note** if $\dim k = 0$ use rank $= \dim \mathbb{Q} \otimes_{\mathbb{Z}} \pi_1(Q,I)$

if $\dim k = p$ use p-rank $= \dim \mathbb{F}_p \otimes_{\mathbb{Z}} \pi_1(Q,I)$

could be bigger if π_1 has p torsion!

Fact max rank $\pi_1(Q,I) \leq$ # of holes in Q $= \left|\begin{smallmatrix}\text{connected}\\ \text{components}\end{smallmatrix}\right| - |Q_0| + |Q_1|$

equality for monomial algebras

You can tell how many "holes" A has from its derived / stable equivalence class

Cor Derived equivalent monomial algebras have the same number of arrows $\longrightarrow$ Compare Avella-Alaminos-Giess gentle alg's and Antipov Zvonoreva Brauer graph algebras.

Final application

A *simply connected* if $\pi_1(Q, I) = 0$ for all presentations $A \cong kQ/I$

Is this equivalent to $HH^1(A) = 0$?

Yes in lots of cases

but $\exists$ counterexample due to Buchweitz Liu .

Buchweitz Liu,
Cohelo –
Lanzilotta –
Savioli
Assem Lanzilotta
Le Meur
Bustamante

Cor A is simply connected $\iff$ $HH^1(A)$ has no tori

In particular, being simply connected is closed under SEMT

uses both parts