THE CATEGORY OF LOCAL REPRESENTATIONS OF A FINITE GROUP

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The category of local representations of a finite group

Based on joint work with

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Isle of Skye, 2015

Inspiration for this project:

n^{th} Morava K-theory

structure of the $K(n)$-local category

of spectra in stable homotopy theory

Setting for this talk

or a finite group scheme

G a finite group, k a field of characteristic $p>0$

kG = group algebra (self-injective algebra)

$H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ group cohomology

(graded comm. noetherian algebra)

$\mathrm{Mod}\,kG$ = cat. of all kG-modules

$\mathrm{StMod}\,kG$ = stable category (modulo projectives)

- compactly generated triangulated category
- admits a tensor product $- \otimes_k -$ (diagonal action)

  and function objects $\mathrm{Hom}_k(-,-)$

$\rightsquigarrow$ tensor triangulated category

— " —

$\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$ = set of homogeneous prime ideals different from $H^{>0}(G,k)$

$\Gamma_{\mathfrak{p}} : \mathrm{StMod}\,kG \longrightarrow \mathrm{StMod}\,kG$ local cohomology functor

$X \longmapsto X \otimes (\Gamma_{\mathfrak{p}} k)$ (exact and idempotent, preserving all $\oplus$s)

Rickard idempotent module

$\Gamma_p(\operatorname{StMod} kG)$ = minimal tensor ideal localising subcategory of $\operatorname{StMod} kG$

(= category of local representations of G)

Aim of this talk Discuss the structure of $\Gamma_p(\operatorname{StMod} kG)$ as a tensor triangulated category (= smallest building block of $\operatorname{StMod} kG$).

Plan of this talk

- an analogy (highest weight categories)
- the local cohomology functor Γ_p
- compact / dualising objects in $\Gamma_p(\operatorname{StMod} kG)$
- an example (Klein four group)

Digression FD Seminar:

finite dimensional algebras

versus finite dimensional representations

Highest weight categories via recollements

Recall: a recollement of abelian/triangulated categories is a diagram of functors

$$\begin{tikzcd} \mathcal{C}' \arrow[r, tail, "i" description] & \mathcal{C} \arrow[l, bend right, two heads, "i_\lambda"'] \arrow[l, bend left, two heads, "i_q"] \arrow[r, two heads, "p" description] & \mathcal{C}'' \arrow[l, bend right, tail, "p_\lambda"'] \arrow[l, bend left, tail, "p_q"] \end{tikzcd}$$

- $\operatorname{Ker} p = \operatorname{Im} i$
- (i_λ, i, i_q) and (p_λ, p, p_q) are adjoint triples
- i, p_λ, p_q are fully faithful

$\mathcal{A}$ an (abelian) length category

$\{L_\lambda\}_{\lambda \in \Lambda}$ a representative set of simple objects

$\Lambda = (\Lambda, \leq)$ finite poset (weights)

$\mathcal{A} \ni X \mapsto \operatorname{Supp} X := \{\lambda \in \Lambda \mid L_\lambda \text{ compos. factor of } X\}$

support

For $\mathcal{U} \subseteq \Lambda$

$$\mathcal{A}_{\mathcal{U}} := \{X \in \mathcal{A} \mid \operatorname{Supp} X \subseteq \mathcal{U}\} \subseteq \mathcal{A}$$

(Serre subcategory)

Suppose: There is a proj. cover $\Delta_\lambda \to L_\lambda$ in $\mathcal{A}_{\leq\lambda}$ $\lambda \in \Lambda$.

Cline-Parshall-Scott, 1988

<u>Theorem</u> $\mathcal{A}$ is a highest weight category with standard objects $\{\Delta_\lambda\}_{\lambda\in\Lambda}$ $\Longleftrightarrow$

For each $\lambda \in \Lambda$ there is a recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}_{<\lambda} \arrow[r] & \mathcal{A}_{\leq\lambda} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, "p"] & \operatorname{mod} K_\lambda \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

with $p = \operatorname{Hom}(\Delta_\lambda, -)$ and $K_\lambda = \operatorname{End}(\Delta_\lambda)$ a division ring, inducing a recollement of derived categories

$$\begin{tikzcd} D^b(\mathcal{A}_{<\lambda}) \arrow[r] & D^b(\mathcal{A}_{\leq\lambda}) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & D^b(\operatorname{mod} K_\lambda) \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}.$$

<u>Idea</u> Standard objects Δ_λ are building blocks of $\mathcal{A}$, glued via localisation sequences

$$\operatorname{Filt}\{\Delta_\mu \mid \mu < \lambda\} \rightleftarrows \operatorname{Filt}\{\Delta_\mu \mid \mu \leq \lambda\} \rightleftarrows \operatorname{Filt}\{\Delta_\lambda\}.$$

(given by exact functors)

$\operatorname{Filt}(\mathcal{X}) :=$ smallest extension closed subcategory containing $\mathcal{X}$

Representations of finite groups via recollements (analogy)

$\mathrm{StMod}\, kG \ni X \longmapsto \mathrm{supp}\, X := \{ \mathfrak{p} \in \mathrm{Proj}\, H^*(G,k) \mid \Gamma_{\mathfrak{p}} X \neq 0 \}$

support

$\mathrm{Proj}\, H^*(G,k)$ poset (ordered by inclusion)

Benson - Iyengar - K - Pevtsova, 2017

Theorem For each $\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$ there is a recollement

functor $\Gamma_{\mathfrak{p}}$

$$(\mathrm{StMod}\, kG)_{<\mathfrak{p}} \rightleftarrows (\mathrm{StMod}\, kG)_{\leq \mathfrak{p}} \rightleftarrows \Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG)$$

Goal Identify the local building blocks (analogues of standard objects Δ_λ).

Local cohomology

$R = H^*(G, k)$ gr. comm. Noeth.

$\mathcal{T} =$ StMod kG comp. gen. triang. cat.

$X \in \mathcal{T}$ compact : $\mathrm{Hom}(X, -)$ preserves all $\bigoplus$s

$\mathcal{T}^c = \{X \in \mathcal{T} \mid X \text{ compact}\}$ thick subcat.

$X, Y \in \mathcal{T}$ (equivalent to stmod kG)

$\mathrm{Hom}^*(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(X, \Sigma^n Y)$ graded R-module

via $R \xrightarrow[-\otimes X]{} \mathrm{End}^*(X)$ or $R \longrightarrow \mathrm{End}^*(Y)$

Fix $p \in \operatorname{Spec} R$

$X \in \mathcal{T}$ p-local if $\mathrm{Hom}^*(C, X) \xrightarrow{\sim} \mathrm{Hom}^*(C, X)_p$ $\forall$ compact C

$\mathcal{T}_p = \{X \in \mathcal{T} \mid X \text{ } p\text{-local}\} \hookrightarrow \mathcal{T}$

admits a left adjoint $X \mapsto X_p$

$\underline{a} \subseteq R$ ideal

R-module M is $\underline{a}$-torsion if $M_q = 0$ for $\underline{a} \not\subseteq q$

$X \in \mathcal{J}$ is $\underline{a}$-torsion if $\mathrm{Hom}^*(C, X)$ is $\underline{a}$-torsion R-module

$$\Gamma_{V(a)} \mathcal{J} = \{ X \in \mathcal{J} \mid X \text{ is } \underline{a}\text{-torsion} \} \hookrightarrow \mathcal{J}$$

has a right adjoint $X \mapsto \Gamma_{V(a)} X$

$$\Gamma_{\mathfrak{p}} \mathcal{J} := \{ X \in \mathcal{J} \mid X \ \mathfrak{p}\text{-local and } \mathfrak{p}\text{-torsion} \}$$

$$\Gamma_{\mathfrak{p}} : \mathcal{J} \longrightarrow \mathcal{J}, \quad X \longmapsto \Gamma_{V(\mathfrak{p})}(X_{\mathfrak{p}})$$

$$\begin{tikzcd}
\mathcal{J}^c \arrow[r, two heads, "(\ )_{\mathfrak{p}}"] & (\mathcal{J}_{\mathfrak{p}})^c & \\
\mathcal{J} \arrow[r, two heads, shift left] & \mathcal{J}_{\mathfrak{p}} \arrow[l, hook, shift left] \arrow[r, shift right, "\Gamma_{V(\mathfrak{p})}"'] & \Gamma_{\mathfrak{p}} \mathcal{J} \arrow[l, hook, shift right]
\end{tikzcd}$$

Compact objects ?

$$(\Gamma_{\mathfrak{p}} \mathcal{J})^c = \Gamma_{\mathfrak{p}} \mathcal{J} \cap (\mathcal{J}_{\mathfrak{p}})^c$$

<u>Problem</u> $\Gamma_{\mathfrak{p}} k$ (tensor unit) in $\Gamma_{\mathfrak{p}} \mathcal{J}$ <u>not</u> compact except when $\mathfrak{p}$ minimal

Compact / dualising objects in $T_p(\mathrm{StMod}\, kG)$

$\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ tensor triang. and comp. gen.

$\mathcal{H}om(X, -)$ right adj. of $X \otimes -$ (Brown repres.)

$X \longmapsto \mathcal{H}om(-, \mathbb{1})$ Spanier-Whitehead duality

also right

<u>X dualising</u>: $\mathcal{H}om(X, \mathbb{1}) \otimes Y \xrightarrow{\sim} \mathcal{H}om(X, Y)$

<u>**Lemma**</u> For $\mathcal{T} = \mathrm{StMod}\, kG$: compact = dualising

Same for $\mathcal{T}_p$, $p \in \mathrm{Proj}\, R$.

<u>**Theorem**</u> (BIKP, 2020) For $X \in \Gamma_p(\mathcal{T})$ are equivalent:

(1) $\mathrm{Hom}^*(C, X)$ is a finite length R_p-module for all $C \in (\Gamma_p \mathcal{T})^c$

(2) $\mathrm{Hom}^*(C, X)$ is an artinian R_p-module for all $C \in \mathcal{T}^c$

(3) X is a direct summand of $T^p C$ for some $C \in \mathcal{T}^c$

(4) X is a dualising object in $T_p\mathcal{T}$

<u>Corollary</u> These objects form a thick subcat. of $T_p(\mathrm{StMod}\,kG)$ closed under $-\otimes-$, $\mathcal{H}om(-,-)$ and SG-duality. Moreover, it is a Krull-Schmidt category (consisting of Σ-pure-inj kG-modules).

<u>Remarks</u> 1) $X \in T_p\mathcal{T}$ compact iff $\mathrm{Hom}^*(C, X)$ finite length over R_p $\forall C \in \mathcal{T}^c$

2) Proof uses Gorenstein property of StMod kG, Matlis duality, Brown repres., StMod kG is strongly generated

3) There is an analogue for $D(\mathrm{Mod}\,A)$, A comm. noeth.

Example Klein four group $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

char $k = 2$

$$kG \cong k[x,y]/(x^2, y^2)$$

$$\bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet \qquad x \circlearrowleft \bullet \circlearrowright y \Big/ \begin{matrix} x^2, y^2 \\ xy - yx \end{matrix}$$

$$X: \; V \rightrightarrows W \longmapsto \bar{X}: \; \circlearrowleft V \oplus W \circlearrowright$$

Kronecker repres. group repres.

$$H^*(G,k) \cong k[\zeta_0, \zeta_1] \qquad \zeta_i \in \operatorname{Ext}^1(k,k)$$

$$\operatorname{Proj} H^*(G,k) \cong \mathbb{P}^1_k$$

$\mathfrak{p} = (0)$ generic point

$$Q: k(t) \underset{t}{\overset{1}{\rightrightarrows}} k(t) \longmapsto \bar{Q}: \; \circlearrowleft k(t) \oplus k(t) \circlearrowright$$

generic repres.

$$\Gamma_{\mathfrak{p}}(\operatorname{StMod} kG) = \operatorname{Add} \bar{Q}$$

$\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$ closed point

tube $\{R_{\mathfrak{p}}[n] \mid n \geq 1\}$ $\quad R_{\mathfrak{p}}[\infty] = \varinjlim R_{\mathfrak{p}}[n]$

regular repres. Prüfer module

$$\Gamma_p(\mathrm{StMod}\, kG)^c = \mathrm{add}\left\{ \overline{R}_p[n] \;\middle|\; 1 \le n < \infty \right\}$$ compacts

$$\Gamma_p(\mathrm{stmod}\, kG) = \mathrm{add}\left\{ \overline{R}_p[n] \;\middle|\; 1 \le n \le \infty \right\}$$ dualising objects

Conclusion: $\Gamma_p(\mathrm{stmod}\, kG)$ is a completion of $\Gamma_p(\mathrm{StMod}\, kG)^c$!