Finite-dimensional differential graded algebras and their properties

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Introduction

The main goal of my talk is to describe the world of derived noncommutative (DN) schemes, which are related to finite-dimensional differential graded $\mathbb{k}$–algebras. (We fix a base field $\mathbb{k}$.)

Plan:

- Differential graded algebras and modules
- Derived noncommutative schemes and their properties
- Gluing of DN schemes and finite-dimensional DGAs
- Radicals for finite-dimensional DGAs and (semi)simplicity
- Auslander construction for finite-dimensional DGAs
- Description of DN schemes for finite-dimensional DGAs
- Geometric realizations of finite-dimensional DGAs
- Geometric description of DN schemes for finite-dimensional DGAs
- Examples

Differential graded algebras and modules

A **differential graded $\Bbbk$–algebra (=DGA)** $\mathscr{R} = (R, d_{\mathscr{R}})$ is a $\mathbb{Z}$–graded $\Bbbk$-algebra

$$R = \bigoplus_{q\in\mathbb{Z}} R^q$$

with a differential $d_{\mathscr{R}} : R \to R$ (degree 1 and $d_{\mathscr{R}}^2 = 0$) that satisfies Leibniz rule

$$d_{\mathscr{R}}(xy) = d_{\mathscr{R}}(x)\, y + (-1)^q x\, d_{\mathscr{R}}(y), \quad \text{for all} \quad x \in R^q, y \in R.$$

A **DG module** M over $\mathscr{R}$ is a graded right R-module $M = \bigoplus_q M^q$ with a differential $d_{\mathrm{M}} : M \to M$ of degree 1 for which $d_{\mathrm{M}}^2 = 0$ and

$$d_{\mathrm{M}}(mr) = d_{\mathrm{M}}(m)\, r + (-1)^q m\, d_{\mathscr{R}}(r), \quad \text{for all} \quad m \in M^q, r \in R.$$

For two M, N define a complex $\mathrm{Hom}_{\mathscr{R}}(\mathrm{M}, \mathrm{N})$ as a graded vector space

$$\mathrm{Hom}_R^{gr}(M, N) := \bigoplus_{q\in\mathbb{Z}} \mathrm{Hom}_R(M, N)^q,$$

with the differential D acting as

$$D(f) = d_{\mathrm{N}} \circ f - (-1)^q f \circ d_{\mathrm{M}} \quad \text{for each} \quad f \in \mathrm{Hom}_R(M, N)^q.$$

Categories of differential graded modules

Let $\mathscr{M}od\text{–}\mathscr{R}$ be the DG category of right DG $\mathscr{R}$–modules and let $\mathscr{A}c\text{–}\mathscr{R}$ be the full DG subcategory of acyclic DG modules.

The **derived category** $\mathcal{D}(\mathscr{R})$ is defined as $\mathcal{H}^0(\mathscr{M}od\text{–}\mathscr{R})/\mathcal{H}^0(\mathscr{A}c\text{–}\mathscr{R})$.

A DG module is called **free** if it is isomorphic to a direct sum of $\mathscr{R}[m]$.

A DG module P is **semi-free** if it has a filtration $0 = \Phi_0 \subset \Phi_1 \subset \ldots = \mathrm{P}$ with free quotient Φ_{i+1}/Φ_i. The full DG subcategory of semi-free DG modules is denoted by $\mathscr{SF}\text{–}\mathscr{R}$.

The inclusion $\mathscr{SF}\text{–}\mathscr{R} \hookrightarrow \mathscr{M}od\text{–}\mathscr{R}$ induces $\mathcal{H}^0(\mathscr{SF}\text{–}\mathscr{R}) \xrightarrow{\sim} \mathcal{D}(\mathscr{R})$.

Denote by $\mathscr{SF}_{fg}\text{–}\mathscr{R} \subset \mathscr{SF}\text{–}\mathscr{R}$ the full DG subcategory of f.g. semi-free DG modules, i.e. $\Phi_n = \mathrm{P}$, and Φ_{i+1}/Φ_i is a finite direct sum of $\mathscr{R}[m]$.

A **DG category of perfect modules** $\mathscr{P}erf\text{–}\mathscr{R} \subset \mathscr{SF}\text{–}\mathscr{R}$ consists of all DG modules that are h-equi to direct summands of object of $\mathscr{SF}_{fg}\text{–}\mathscr{R}$.

The homotopy category $\mathcal{H}^0(\mathscr{P}erf\text{–}\mathscr{R})$ is denoted by $\mathrm{perf}\text{–}\mathscr{R}$. It is equi to the tri-subcategory of compact objects $\mathcal{D}(\mathscr{R})^c \subset \mathcal{D}(\mathscr{R})$.

Derived noncommutative schemes and their properties

A **derived noncommutative (DN) scheme** $\mathscr{X}$ over $\Bbbk$ is a $\Bbbk$–linear DG category of the form $\mathscr{P}\!erf\!-\!\mathscr{R}$, where $\mathscr{R}$ is a coh-ly bounded DGA.

The category $\operatorname{perf}\!-\!\mathscr{R}$ is called the category of perfect complexes on $\mathscr{X}$, while $\mathcal{D}(\mathscr{R})$ is called the derived category of q-coh sheaves on $\mathscr{X}$.

If X is a q-comp and q-sep scheme, then $\mathscr{P}\!erf\!-\!X$ is q-equi to $\mathscr{P}\!erf\!-\!\mathscr{R}$ for a coh-ly bounded DGA $\mathscr{R}$ (A.Neeman; A.Bondal & M.Van den Bergh).

A DN scheme $\mathscr{X} = \mathscr{P}\!erf\!-\!\mathscr{R}$ is **proper** iff the cohomology algebra $\bigoplus_{p\in\mathbb{Z}} H^p(\mathscr{R})$ is f.d. It is a property of $\mathscr{P}\!erf\!-\!\mathscr{R}$.

A DN scheme $\mathscr{X} = \mathscr{P}\!erf\!-\!\mathscr{R}$ is called **$\Bbbk$–smooth** if the DGA $\mathscr{R}$ is perfect as the DG bimodule, i.e. as the DG module over $\mathscr{R}^{\circ} \otimes_{\Bbbk} \mathscr{R}$.

V. Lunts and O. Schnürer proved that smoothness is Morita invariant.

Moreover, notions of smoothness and properness are compatible with the corresponding notions for (commutative) separated scheme of finite type.

Thus, any f.d. DGA $\mathscr{R}$ defines a proper DN scheme $\mathscr{X} = \mathscr{P}\!erf\!-\!\mathscr{R}$.

Finite-dimensional DG algebras and Morita equivalence

It is a property of DN scheme $\mathscr{X}$ to be $\mathscr{P}erf\!-\!\mathscr{R}$ with f.d. DGA $\mathscr{R}$.

Proposition: *Let $\mathscr{R}$ be a f.d. DGA. Let $\mathrm{M} \in \mathscr{P}erf\!-\!\mathscr{R}$ be a perfect DG module. Then DGA $\mathrm{End}_{\mathscr{R}}(\mathrm{M})$ is q-iso to a f.d. DGA.*

Corollary: *Let $\mathscr{R}_1, \mathscr{R}_2$ be two DGAs and $\mathscr{P}erf\!-\!\mathscr{R}_1 \cong \mathscr{P}erf\!-\!\mathscr{R}_2$ are q-equi. If $\mathscr{R}_1$ is q-iso to a f.d. DGA, then $\mathscr{R}_2$ is also q-iso to a f.d. DGA.*

Simplest DN schemes have a form $\mathscr{P}erf\!-\!S$, where S are f.d. division or, more general, (semi)simple algebras. Gluing such schemes we obtain a large class of DN schemes with full semi-exceptional collections.

Rem: An object $E \in \mathcal{T}$ is called **semi-exceptional** if $\mathrm{Hom}(E, E[l]) = 0$, when $l \neq 0$, and $\mathrm{Hom}(E, E) = S$, where S is a f.d. semisimple algebra. A **semi-exceptional collection** in $\mathcal{T}$ is a sequence $(E_1, \dots, E_n)$ of semi-exceptional objects with $\mathrm{Hom}(E_i, E_j[l]) = 0$ for all l if $i > j$.

A collection is called **full**, if it generates the tri-category $\mathcal{T}$.

Let $\mathscr{R}$ and $\mathscr{S}$ be two DGAs and let T be a DG $\mathscr{S}$-$\mathscr{R}$–bimodule. The **lower triangular** DGA $\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S}$ is defined as follows:

$$\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S} = \begin{pmatrix} \mathscr{R} & 0 \\ \mathsf{T} & \mathscr{S} \end{pmatrix}$$

A **gluing** $\mathscr{X} \underset{\mathsf{T}}{\oplus} \mathscr{Y}$ of $\mathscr{X} = \mathscr{P}\!erf\!-\!\mathscr{R}$ and $\mathscr{Y} = \mathscr{P}\!erf\!-\!\mathscr{S}$ w.r.t a DG $\mathscr{S}$-$\mathscr{R}$–bimodule T is the DG category $\mathscr{P}\!erf\!-\!(\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S})$.

The DG category $\mathscr{P}\!erf\!-\!(\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S})$ gives a DN scheme iff DGA $\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S}$ is coh-ly bounded (i.e. iff T has only finitely many nontrivial cohomology).

The tri-cat $\operatorname{perf}\!-\!(\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S})$ has semi-orth decomp $\langle \operatorname{perf}\!-\!\mathscr{R}, \operatorname{perf}\!-\!\mathscr{S}\rangle$.

Moreover, any semi-orthogonal decomposition is induced by a gluing.

Proposition: *Let $\mathscr{E}$ be a DGA. Suppose there is a semi-orth decomp* $\operatorname{perf}\!-\!\mathscr{E} = \langle \mathcal{A}, \mathcal{B}\rangle$. *Then $\mathscr{P}\!erf\!-\!\mathscr{E}$ is q-equi to a gluing $\mathscr{A} \underset{\mathsf{T}}{\oplus} \mathscr{B}$.*

$\mathscr{A}, \mathscr{B}$ have form $\mathscr{P}\!erf\!-\!\mathscr{R}, \mathscr{P}\!erf\!-\!\mathscr{S}$ and $\mathscr{P}\!erf\!-\!\mathscr{E} \simeq \mathscr{P}\!erf\!-\!(\mathscr{R}\underset{\mathsf{T}}{\llcorner}\mathscr{S})$.

Gluing of finite-dimensional DG algebras

Gluings inherit properties of its components if the glue is good enough.

$\mathscr{X} \bigoplus_{\mathsf{T}} \mathscr{Y}$ is proper $\Leftrightarrow$ $\mathscr{X}, \mathscr{Y}$ are proper and T is coh-ly finite.
$\mathscr{X} \bigoplus_{\mathsf{T}} \mathscr{Y}$ is smooth $\Leftrightarrow$ $\mathscr{X}, \mathscr{Y}$ are smooth and T is perfect bimodule.

Proposition: *Let $\mathscr{R}, \mathscr{S}$ be two f.d. DGAs and T be a perfect bimodule. Then the DGA $\mathscr{R} \sqcup_{\mathsf{T}} \mathscr{S}$ is q-iso to a f.d. DGA.*

Remark: There is an example of gluing of f.d. DGAs via a coh-ly finite bimodule s.t. the resulting DGA is not q-iso to a f.d. DGA (A.Efimov).

Proposition: *Let $\mathscr{R}, \mathscr{S}$ be smooth f.d. DGAs and T be a coh-ly finite bimodule. Then the DGA $\mathscr{R} \sqcup_{\mathsf{T}} \mathscr{S}$ is smooth and it is q-iso to a f.d. DGA.*

Corollary: *Let $\mathscr{R}$ be a DGA. Suppose $\mathrm{perf}\!-\!\mathscr{R}$ is proper and has a full sep. semi-exceptional coll. Then $\mathscr{R}$ is smooth and is q-iso to a f.d. DGA.*

Finite-dimensional DG algebras and radicals

Let $\mathscr{R} = (R, d_{\mathscr{R}})$ be a finite-dimensional DGA over a field $\Bbbk$. Let $\underline{R}$ be the underlying ungraded algebra of R. Denote by $J = J(\underline{R}) \subset \underline{R}$ the (Jacobson) radical of $\underline{R}$. We know that $J^n = 0$ for some n.

The radical $J \subset R$ is a graded ideal. On the other hand, the radical J is not necessary a DG ideal, i.e. $d_{\mathscr{R}}(J)$ is not necessary a subspace of J.

Let $I \subset R$ be a graded (two-sided) ideal. An **internal** DG ideal $\mathsf{I}_{-} = (I_{-}, d_{\mathscr{R}})$ consists of all $r \in I$ such that $d_{\mathscr{R}}(r) \in I$.

An **external** DG ideal $\mathsf{I}_{+} = (I_{+}, d_{\mathscr{R}})$ is the sum $I + d_{\mathscr{R}}(I)$.

Lemma: *The natural homomorphism* $\mathscr{R}/\mathsf{I}_{-} \to \mathscr{R}/\mathsf{I}_{+}$ *is a q-iso.*

The DG ideals $\mathsf{J}_{-}, \mathsf{J}_{+}$ are called internal and external DG radicals of $\mathscr{R}$.

Useful **properties** of J_{-} and J_{+} :

- the DG ideal J_{-} is nilpotent,
- the underline algebra $\underline{R}/\mathsf{J}_{+}$ is semisimple.

Semisimple finite-dimensional DG algebras

A f.d. DGA $\mathscr{S}$ is called **simple**, if the DG category $\mathscr{P}erf\!-\!\mathscr{S}$ is q-equi to $\mathscr{P}erf\!-\!D$, where D is a f.d. division algebra. It is **semisimple**, if $\mathscr{P}erf\!-\!\mathscr{S}$ is a gluing $\mathscr{P}erf\!-\!D_1 \oplus \cdots \oplus \mathscr{P}erf\!-\!D_m$. In addition, it is called **separable**, if all division algebras D_i are separable over $\Bbbk$.

A f.d. DGA $\mathscr{S} = (S, d_{\mathscr{S}})$ is called **abstractly (semi)simple**, if the algebra $\underline{S}$ is (semi)simple. (It is not invariant w.r.t q-iso of DGA.)

Any (semi)simple DGA is q-iso to abstractly (semi)simple DGA. Indeed, for any $\mathrm{N} \in \mathscr{P}erf\!-\!D$ the DGA $\mathrm{End}_D(\mathrm{N})$ is abstractly simple. If $\mathscr{P}erf\!-\!\mathscr{S}$ is q-equi to $\mathscr{P}erf\!-\!D$, then the DGA $\mathscr{S}$ is q-iso to $\mathrm{End}_D(\mathrm{N})$ for some $\mathrm{N} \in \mathscr{P}erf\!-\!D$.

Proposition: *Let $\mathscr{S} = (S, d_{\mathscr{S}})$ be an abstractly simple DGA. Then it is simple and it is iso to $\mathrm{End}_D(\mathrm{N})$ for some $\mathrm{N} \in \mathscr{P}erf\!-\!D$.*

A similar statement holds for abstractly semisimple DGAs.

Thus, the DGAs $\mathscr{R}/\mathrm{J}_+$ and $\mathscr{R}/\mathrm{J}_-$ are semisimple.

Auslander construction for DG algebras I

Let $\mathscr{R} = (R, d_{\mathscr{R}})$ be a f.d. DGA. Denote by J_p the DG ideals $(J^p)_{-}$.
Thus, we obtain a chain of DG ideals $\mathrm{J}_{-} = \mathrm{J}_1 \supseteq \cdots \supseteq \mathrm{J}_n = 0$.

Consider DG modules $\mathrm{M}_p = \mathscr{R}/\mathrm{J}_p$ with $p = 1, \ldots, n$. In particular, $\mathrm{M}_1 \cong \mathscr{R}/\mathrm{J}_{-}$ and $\mathrm{M}_n \cong \mathscr{R}$.

Denote by $\mathscr{E} = (E, d_{\mathscr{E}})$ the DGA $\mathrm{End}_{\mathscr{R}}(\mathrm{M})$, where $\mathrm{M} = \bigoplus_{p=1}^{n} \mathrm{M}_p$.

Consider the DG $\mathscr{E}$–module $\mathrm{P}_n = \mathrm{Hom}_{\mathscr{R}}(\mathrm{M}, \mathrm{M}_n)$. It is h-projective and $\mathrm{End}_{\mathscr{E}}(\mathrm{P}_n) \cong \mathscr{R}$. Thus, the DG $\mathscr{E}$–modules P_n is a DG $\mathscr{R}$-$\mathscr{E}$–bimodule and it induces two adjoint functors

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{R}} \mathrm{P}_n : \mathcal{D}(\mathscr{R}) \to \mathcal{D}(\mathscr{E}), \quad \mathbf{R}\mathrm{Hom}_{\mathscr{E}}(\mathrm{P}_n, -) : \mathcal{D}(\mathscr{E}) \to \mathcal{D}(\mathscr{R})$$

The DG $\mathscr{E}$–module P_n is perfect and, hence, we have a functor

$$(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{R}} \mathrm{P}_n : \mathrm{perf}\text{–}\mathscr{R} \longrightarrow \mathrm{perf}\text{–}\mathscr{E}.$$

Auslander construction for DG algebras II

The following theorem allows us to describe the DG category $\mathscr{P}erf\text{–}\mathscr{E}$ and a relation between DG categories $\mathscr{P}erf\text{–}\mathscr{E}$ and $\mathscr{P}erf\text{–}\mathscr{R}$.

Theorem: *Let* $\mathscr{R} = (R, d_{\mathscr{R}})$ *be a f.d. DGA of index nilpotency* n *and let* $\mathscr{E} = \mathrm{End}_{\mathscr{R}}(\bigoplus_{p=1}^{n} \mathrm{M}_p)$ *be as above. Then:*

1) $(-)\overset{\mathbf{L}}{\otimes}_{\mathscr{R}} \mathrm{P}_n : \mathrm{perf}\text{–}\mathscr{R} \to \mathrm{perf}\text{–}\mathscr{E}, \quad \mathcal{D}(\mathscr{R}) \to \mathcal{D}(\mathscr{E})$ *are fully faithful.*
2) *The category* $\mathrm{perf}\text{–}\mathscr{E}$ *has a full semi-exceptional collection.*
3) *If* $\mathscr{S}_+ = \mathscr{R}/\mathrm{J}_+$ *is separable, then the DGA* $\mathscr{E}$ *is smooth.*
4) *If* $\mathscr{R}$ *is smooth, then* $\mathrm{perf}\text{–}\mathscr{R}$ *is admissible in* $\mathrm{perf}\text{–}\mathscr{E}$.

The DGAs $\mathscr{R}$ and $\mathscr{E}$ define two proper DN schemes $\mathscr{X} = \mathscr{P}erf\text{–}\mathscr{R}$ and $\mathscr{Y} = \mathscr{P}erf\text{–}\mathscr{E}$. The bimodule P_n gives a morphism $\mathrm{f} : \mathscr{Y} \to \mathscr{X}$. Since $\mathscr{Y}$ is smooth, the morphism f is a resolution of sing. for $\mathscr{X}$.

The following corollary provides a complete characterization of all DN schemes $\mathscr{X} = \mathscr{P}erf\text{–}\mathscr{R}$ for a f.d. DGA $\mathscr{R}$ with separable $\mathscr{R}/\mathrm{J}_+$.

Description of DN schemes for finite-dimensional DGAs

Corollary: *Let* $\mathscr{A}$ *be a small* $\Bbbk$*–linear DG category. The following properties are equivalent:*

1) $\mathscr{A}$ *is q-equi to* $\mathscr{P}erf\!-\!\mathscr{R}$*, where* $\mathscr{R}$ *is a f.d. DGA for which* $\mathscr{R}/\mathrm{J}_+$ *is separable.*
2) $\mathscr{A}$ *is q-equi to a full DG subcategory* $\mathscr{C}$ *of a pretriangulated* $\Bbbk$*–linear DG category* $\mathscr{B}$ *such that*
   a) $\mathcal{H}^0(\mathscr{B})$ *is a proper idempotent complete* tri*-category possessing a full sep. semi-exceptional collection,*
   b) $\mathcal{H}^0(\mathscr{C}) \subseteq \mathcal{H}^0(\mathscr{B})$ *is an idempotent complete* tri*-subcategory admitting a classical generator.*

Moreover, the DGA $\mathscr{R}$ *is smooth iff* $\mathcal{H}^0(\mathscr{C}) \subseteq \mathcal{H}^0(\mathscr{B})$ *is admissible.*

Rem: $\mathcal{T}$ is **idempotent complete** if it contains kernels of all projectors. A set $S \subset \mathcal{T}$ **classically generates** tri-cat $\mathcal{T}$ if the smallest idempotent complete full tri-subcat of $\mathcal{T}$, which contains S, coincides with $\mathcal{T}$.

Corollary: *Let* $\mathscr{R}$ *be a smooth f.d. DGA. Then* $K_0(\mathrm{perf}\!-\!\mathscr{R}) \cong \mathbb{Z}^r$.

Geometric realizations

A **geometric realization** of a DN scheme $\mathscr{X} = \mathscr{P}erf\text{-}\mathscr{R}$ is a usual scheme Z and a localizing subcategory $\mathcal{L} \subseteq \mathcal{D}_{\mathsf{Qcoh}}(Z)$, the natural enhancement $\mathscr{L}$ of which is q-equi to $\mathscr{S}\mathscr{F}\text{-}\mathscr{R}$.

The most important class of geometric realizations is related to q-fun $\mathsf{F}: \mathscr{P}erf\text{-}\mathscr{R} \to \mathscr{P}erf\text{-}Z$ for which $\mathrm{perf}\text{-}\mathscr{R} \hookrightarrow \mathrm{perf}\text{-}Z$ is fully faithful.

Let X and Y be two usual irreducible smooth projective schemes and let $\mathsf{E} \in \mathscr{P}erf\text{-}(X \times_{\Bbbk} Y)$. Consider the DN scheme $(\mathscr{P}erf\text{-}X) \underset{\mathsf{E}}{\oplus} (\mathscr{P}erf\text{-}Y)$ and denote it by $\mathscr{Z} := X \underset{\mathsf{E}}{\oplus} Y$. The DN scheme $\mathscr{Z}$ is smooth and proper, but it is not equi to a usual commutative scheme, in general.

Theorem: *Let* $\mathscr{Z}$ *be as above. Then there is a geometric realization* $\mathsf{F}: \mathscr{P}erf\text{-}\mathscr{Z} \hookrightarrow \mathscr{P}erf\text{-}V$, *where* V *is a usual smooth projective scheme.*

The proof is constructive and in this case $\mathrm{perf}\text{-}V$ has a semi-orth decomp $\mathrm{perf}\text{-}V = \langle \mathcal{N}_1, \ldots \mathcal{N}_k \rangle$, where all $\mathcal{N}_i$ are equi to one of the following four categories: $\mathrm{perf}\text{-}\Bbbk$, $\mathrm{perf}\text{-}X$, $\mathrm{perf}\text{-}Y$, and $\mathrm{perf}\text{-}(X \times_{\Bbbk} Y)$.

Geometric realizations of finite-dimensional DGAs

This theorem can be extended to the case of geometric smooth proper DN schemes (i.e. $\mathscr{N} \subset \mathscr{P}erf\!-X$ for which $\mathcal{N} = \mathcal{H}^0(\mathscr{N})$ is admissible in $\mathrm{perf}\!-X$). A gluing $\mathscr{N}_1 \underset{\mathrm{T}}{\oplus} \mathscr{N}_2$ of any such DN schemes via a perfect bimodule T has a geom realization in a smooth projective scheme V.

Applying them to a f.d. DGA $\mathscr{R}$ and to its Auslander DGA $\mathscr{E}$, we obtain

Theorem: *Let $\mathscr{R}$ be a f.d. DGA and $\mathscr{S}_+ = \mathscr{R}/\mathrm{J}_+$ is separable. Then there are a smooth projective scheme Z and $\mathrm{E} \in \mathscr{P}erf\!-Z$ for which*

1) $\mathrm{End}_{\mathscr{P}erf-Z}(\mathrm{E})$ *is q-iso to* $\mathscr{R}$.
2) $\mathscr{P}erf\!-\mathscr{R}$ *is q-equi to a full DG subcategory* $\mathscr{C} \subset \mathscr{P}erf\!-Z$.
3) $\mathrm{perf}\!-Z$ *has a full sep. semi-exceptional collection.*

Moreover, if $\mathscr{R}$ is smooth, then $\mathcal{H}^0(\mathscr{C})$ is admissible in $\mathrm{perf}\!-Z$.

If $\Bbbk = \overline{\Bbbk}$, then $Z = X_n \to \cdots \to X_1 \to \mathbf{pt}$ is a tower of projective bundles (i.e. $X_{p+1} = \mathbb{P}_{X_p}(E_p)$) and $\mathrm{perf}\!-Z$ has a full exceptional collection.

Geometric description of DN schemes for f.d. DGAs

The following corollary gives a description of DN schemes $\mathscr{X} = \mathscr{P}erf\text{–}\mathscr{R}$ for a f.d. DGA $\mathscr{R}$ with separable $\mathscr{R}/\mathrm{J}_+$ in geometric terms.

Corollary: *Let $\mathscr{A}$ be a small $\Bbbk$–linear DG category. The following properties are equivalent:*

1) *$\mathscr{A}$ is q-equi to $\mathscr{P}erf\text{–}\mathscr{R}$, where $\mathscr{R}$ is a f.d. DGA for which $\mathscr{R}/\mathrm{J}_+$ is separable.*
2) *$\mathscr{A}$ is q-equi to a full DG subcategory $\mathscr{C} \subseteq \mathscr{P}erf\text{–}Z$, where*
    a) *Z is a smooth projective scheme with a full sep. semi-exceptional collection.*
    b) *$\mathcal{H}^0(\mathscr{C}) \subseteq \mathrm{perf}\text{–}Z$ is an idempotent complete* tri*-subcategory admitting a classical generator.*

Moreover, the DGA $\mathscr{R}$ is smooth iff $\mathcal{H}^0(\mathscr{C}) \subseteq \mathrm{perf}\text{–}Z$ is admissible.

Again, if $\Bbbk = \bar{\Bbbk}$, then Z can be taken as a tower of projective bundles and in this case $\mathrm{perf}\text{–}Z$ has a full exceptional collection.

Examples

Consider two examples of Auslander construction and geom realizations.

Example 1: Let $\Lambda^* V = \bigoplus_q \Lambda^q V$ be the exterior algebra, where elements of V have degree 1. Denote by $\mathscr{R}$ the DG algebra $\Lambda^* V$ with $d = 0$.

The Auslander construction applied to $\mathscr{R} = (\Lambda^* V, 0)$, with $\deg V = 1$, $\dim V = n$ provides a full embedding $\text{perf}-\mathscr{R}$ to $\text{perf}-\mathbb{P}^n$ sending $\mathscr{R} \in \text{perf}-\mathscr{R}$ to the structure sheaf $\mathcal{O}_p \in \text{perf}-\mathbb{P}^n$ of a point $p \in \mathbb{P}^n$.

Example 2: Let $\mathscr{R}$ be $(\Bbbk[x]/x^n, 0)$, with $\deg x = 1$. Applying the Auslander construction to it, we obtain that the DG category $\mathscr{P}erf-\mathscr{E}$ is q-equi to $\mathscr{P}erf-\Bbbk[Q]$, where $\Bbbk[Q]$ is the path algebra of the quiver

$$Q = \left(\underset{1}{\bullet} \overset{a_1}{\underset{b_1}{\rightrightarrows}} \underset{2}{\bullet} \overset{a_2}{\underset{b_2}{\rightrightarrows}} \underset{3}{\bullet} \cdots\cdots\cdots \underset{n-1}{\bullet} \overset{a_{n-1}}{\underset{b_{n-1}}{\rightrightarrows}} \underset{n}{\bullet} \;\middle|\; a_{i+1}b_i = 0,\ b_{i+1}a_i = 0 \right).$$

The quiver Q appears as a directed Fukaya-Seidel category in an LG model with a superpotential $W : C \to \mathbb{A}^1$ that is a double covering and C is a hyperelliptic curve of genus $g = \left[\frac{n-1}{2}\right]$ without one or two points.