- Cohomology for Drinfeld doubles (aka Drinfeld centers) of finite group schemes 1

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$k = \bar{\mathbb{F}}_p$

Theorem [N, FN18]: Let G be any finite group scheme. Then $Z(\mathrm{rep}\, G) = \mathrm{rep}(\mathrm{Drin\,Doub}\, G)$ has finitely generated cohomology.

- Generic nonsense (fin ⊗-cat)
- Finite group schemes
- The Drinf center/double for G
- Theorem
- History Context
- Methods (defo thry)

- Finite tensor categories (language) 2

For A a Hopf alg consider $\mathrm{rep}(A) = \{\text{fin-dim'l } A\text{-modules}\}$. $\mathrm{rep}(A)$ is monoidal, w/ product $\otimes = \otimes_k$ specified by comult and unit $\mathbb{1} = k$ specified by counit $\epsilon : A \to k$. The antipode provides a duality $V \mapsto V^*$.

Def$^{\underline{n}}$: A <u>finite ⊗-cat</u> $\mathcal{C}$ is a k-linear monoidal cat which is formally indistinguishable from the rep category of a fin dim'l Hopf alg. [$\mathcal{C}$ has fin many simp, enough proj, all obj of fin length, duality $V \mapsto V^*$, etc.]

Ex's · For G a fin disc group, $\mathrm{rep}_k(G) =$ (3)
a finite $\otimes$-cat $\rightsquigarrow$ $\{$fin-dim'l G-rep's$\}$

Familiar (finite) tensor categories

· For $\mathfrak{g}$ a restricted Lie alg

$$\mathrm{rep}^{res}(\mathfrak{g}) = \left\{ \begin{matrix} \text{fin dim'l restricted} \\ \text{rep's of } \mathfrak{g} \quad x^{[p]} \curvearrowright V = (x \cdot -)^p \curvearrowright V \end{matrix} \right\}$$

· For G a finite group scheme

$$\mathrm{Coh}(G) = \left\{ \begin{matrix} \text{Coherent} \\ \text{sheaves on} \\ G \end{matrix} \right\} = \mathrm{rep}(\mathcal{O}(G))$$

$$\supseteq \quad \mathrm{rep}(G)$$

– Reminder on finite group schemes (4)

Recall that a finite group scheme G is the spectrum $G = \mathrm{Spec}(\mathcal{O})$ of a fin-dim'l comm Hopf alg $\mathcal{O}$.

Discrete $\longleftarrow$ - - - - - - $\longrightarrow$ Infinitesimal

S_n, $\mathbb{Z}/d\mathbb{Z}$, D_n, etc. $\qquad$ $G_{(r)}$ (frobenius kernels)

Recall that for any affine group scheme G have p^r-th power map on the alg of fun's. Dually, we have a map of schemes Fr^r called the rth Frobenius

$$1 \to G_{(r)} \to G \xrightarrow{Fr^r} G^{(r)} \to 1$$

This is a faithfully flat map of group schemes, and its kernel $G_{(r)} := \ker(Fr^r)$ is a finite subgroup in G with a single point.

For G a finite group scheme can consider alg of functions $\mathcal{O}(G)$ or the group ring $kG := \mathcal{O}(G)^*$.

Have

$$\mathrm{Coh}(G) = \mathrm{rep}(\mathcal{O}(G))$$

$$\mathrm{rep}(G) = \mathrm{rep}(kG).$$

. . .

- The center $Z(\mathrm{rep}\,G)$ ⑥

For C a finite $\otimes$-cat have fundamental construction, called the Drinfeld Center of C,

$$Z(C) = \left\{ \begin{array}{l} \text{Cat of central} \\ \text{objects in } C \end{array} \right\} = \left\{ \begin{array}{l} \text{Cat of pairs } (V, \xi) \\ \text{w/ } V \text{ in } C \text{ and} \\ \text{a nat isom } \xi : V \otimes - \xrightarrow{\sim} - \otimes V \end{array} \right\}.$$

$Z(C)$ is a fin $\otimes$-cat which is "twice as big" as C, and it controls the morita theory of C. For $\mathrm{rep}(G)$ have elegant interpretation

$$Z(\mathrm{rep}\,G) = \left\{ \begin{array}{l} \text{Cat of } G\text{-equiv} \\ \text{coh shvs on } G \\ \text{under the adj} \\ \text{action } G \overset{\mathrm{adj}}{\curvearrowright} G \end{array} \right\} =: \mathrm{Coh}(G)^G = \mathrm{rep}(\underbrace{\mathcal{O} \rtimes_{\mathrm{ad}} kG}_{\text{called Drinfeld double}}).$$

– Theorem

Thm [N, FN18]: For G any finite group scheme, cohom over $\mathcal{Z}_s = \mathcal{Z}(\mathrm{rep}\, G)$ has the following strong finiteness prop's:

(FG1) For V any obj in $\mathcal{Z}_s$, the extensions $\mathrm{Ext}^\bullet_{\mathcal{Z}}(V,V)$ form a fin gen'd alg, which is fin over its center.

(FG2) For any other W in $\mathcal{Z}_s$, $\mathrm{Ext}^\bullet_{\mathcal{Z}}(V,W)$ is a fin gen'd module over ext's of V on the right and Ext's of W on the left.

(FG3) For all V in $\mathcal{Z}_s$

Always a comm algebra!

$$\mathrm{Kdim}\, \mathrm{Ext}^\bullet_{\mathcal{Z}}(V,V) \leq \mathrm{Kdim}\, \mathrm{Ext}^\bullet_{\mathcal{Z}}(1,1)$$

$$= \mathrm{Kdim}\, \mathrm{Ext}^\bullet_{G}(1,1) + \mathrm{embed.dim}(G).$$

"Known"

Thm [FN18]: For $\mathbb{G}$ a smooth alg group, $G = \mathbb{G}_{(r)}$ an r-th Frob kernel, and $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})$, have

"Known"

$$(*) \quad \mathrm{Spec}\, \mathrm{Ext}^\bullet_{\mathcal{Z}}(1,1) \to \mathrm{Spec}\, \mathrm{Ext}^\bullet_{G}(1,1) \times \mathfrak{g}^{(r)}.$$

Furthermore, $(*)$ is a homeomorphism when $\mathbb{G}$ is, for example, when $\mathbb{G}$ is almost-simple / k and p is sufficiently large.

... But, where are these results coming from...

– Same context

Lets say finite $\otimes$-cat $\mathcal{C}$ is of finite type [over the base field k] if its cohom satisfies the strong fin gen conditions (FG1)–(FG3).

Historically:

- [Evens, Golod, Venkov 60's] For G a discrete finite group, the rep cat rep(G) is of finite type
- [Friedlander-Suslin '97] For G a fin. group scheme, rep(G) of finite type. Also, [Fr-Pevtsova, SFBendel] can describe the spectrum of cohom as a certain moduli space of "special nilpotent elements" in kG. (a nil-cone).
- [Drupieski '10] rep(SG) for a super group scheme ...

[Many examples in char 0, which I won't recall]

Have Conj [N-Plavnik]: If $\mathcal{C}$ of finite type $\Rightarrow Z(\mathcal{C})$ of finite type. Furthermore, in char 0,

$$\mathrm{Kdim}(\mathrm{Cohom}\, Z(\mathcal{C})) = 2\, \mathrm{Kdim}(\mathrm{Cohom}\, \mathcal{C}).$$

Conj [Etingof-Ostrik '05] Any finite $\otimes$-category is in fact of finite type.

Remark: The scheme Spec(Cohom $\mathcal{C}$) should determine "the global structure" of the derived category. Furthermore, Spec(cohom $\mathcal{C}$) should have an interpretation in terms of a 4-dim field theory, det by $\mathcal{C}$ (when $\mathcal{C}$ is ...).

- Methods (defo theory)

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Take again $\mathcal{Z} = \mathcal{Z}(\mathrm{rep}\, G)$ G a finite group scheme.
Let's just consider extensions of the unit $\mathbb{1}$.

Have

$$\mathrm{rep}\, G \xrightarrow[\text{supp at } 1]{\text{sheaves}} \mathcal{Z} = \mathrm{Coh}(G)^G$$

$$\Longrightarrow \quad \mathrm{Ext}^{\bullet}_{G}(\mathbb{1},\mathbb{1}) \longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{Z}}(\mathbb{1},\mathbb{1})$$

This gives a <u>part</u> of cohom, contributed from rep(G).
We need a contribution from Coh(G), or $\mathcal{O}(G)$, in order to fully grasp Cohom for $\mathcal{Z}$.

We employ deformation theory for this.

Consider $G = G_{(r)}$. Have

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$$\begin{tikzcd} G_{(r)} \arrow[r] & G \arrow[d, "\mathrm{Fr}"', "\text{flat}"] \\ & G^{(r)} \end{tikzcd} \qquad (*)$$

realizing G as a flat family of schemes/alg's over $G^{(r)}$ which deforms $G_{(r)}$.

$$\Longrightarrow \quad A_{\text{defo}} := \mathrm{Sym}(\mathfrak{g}^{*(r)}) \longrightarrow \mathrm{Ext}^{\bullet}_{\mathrm{Coh}(G)}(\mathbb{1},\mathbb{1})$$

$\mathfrak{g}^{*(r)} = T_1 G^{(r)}$

cohom deg 2.

[Standard defo theory, Gerstenhaber]

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But this deformation is in fact <u>equivariant</u> in the sense that the adj action of $\mathbb{G}_{(r)}$ on $\mathbb{G}$ restricts to an action on the fibers over $\mathbb{G}^{(r)}$

$$\begin{tikzcd} \mathbb{G}_{(r)} \arrow[r] \arrow[loop below, "\mathrm{adj}\ \mathbb{G}_{(r)}"] & \mathbb{G} \arrow[dr, "Fr"] \arrow[loop below, "\mathrm{adj}\ \mathbb{G}_{(r)}"] & \\ & & \mathbb{G}^{(r)} \end{tikzcd}$$

So $\mathbb{G}$ deforms $\mathbb{G}_{(r)}$ as a $\mathbb{G}_{(r)}$-scheme

$\Longrightarrow$

$$\mathcal{A}_{defo} = \mathrm{Spec}\left(\mathfrak{g}^{*(r)}\right) \longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{Z}}(1,1).$$

Thm [FN]:
The product map $\mathrm{Ext}^{\bullet}_{G}(1,1) \otimes \mathcal{A}_{defo} \xrightarrow{\text{finite}} \mathrm{Ext}^{\bullet}_{\mathcal{Z}}(1,1)$

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The general case is not like this...

Can embed G an arb. fin group scheme into smooth $\mathbb{G}$ and obtain a defo

$$G \longrightarrow \mathbb{G} \xrightarrow{\text{flat}} \mathbb{G}/G.$$

But the fibers in $\mathbb{G}$ are permuted by the adj action of G.

This is because G <u>not</u> normal in $\mathbb{G}$ in general, and

So acts on param space G/G.

But, can consider a new kind of "equiv defo thy"

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Here we allow G to act on the param space, and look for higher deformation classes

$$\mathcal{A}_{\mathrm{defo}}^{\mathrm{higher}} \longrightarrow \mathrm{Ext}^{\bullet}_{Z}(1,1)$$

some (fg) alg w/ generators in high degree

Thm [N]: We can in fact do this, and the prod map

$$\mathrm{Ext}^{\bullet}_{G}(1,1) \otimes \mathcal{A}_{\mathrm{defo}}^{\mathrm{higher}} \longrightarrow \mathrm{Ext}^{\bullet}_{Z}(1,1)$$

is again finite.

—THX—