

- Cohomology for Drinfeld doubles (aka Drinfeld centers) of finite group schemes
- C. Nagar w/ F. Friedlander $\zeta = \overline{F_p}$

Theorem [N, FN18]: Let G be any finite group scheme. Then $Z(\text{rep}(G))$ has finitely generated cohomology.

$\text{rep}(\text{Drinf Doub } G)$

- General nonsense (\mathcal{C} is \mathbb{Q} -cat)
- Finite group schemes
- The Drinf center/double for G
- Theorem
- History Context
- Methods (categorical)

- Finite tensor categories (language) (2)

For A a Hopf alg consider $\text{rep}(A) = \{ \text{fin-dim } A\text{-modules} \}$. $\text{rep}(A)$ is monoidal, w/ product $\otimes = \otimes_A$ specified by comult and unit $\mathbb{I} = \mathbb{1}$ specified by counit $\epsilon: A \rightarrow \mathbb{1}$. The antipode provides a duality $V \mapsto V^*$.

Def: A finite \mathbb{Q} -cat \mathcal{C} is a \mathbb{K} -linear monoidal cat which is formally indistinguishable from the rep category of a fin dim Hopf alg. [\mathcal{C} has fin many simp, enough proj, all obj of fin length, duality $V \mapsto V^*$, etc.]

Finite (finite) tensor categories

Ex's • For G a finite group, $\text{rep}_k(G) =$ ③
 or finite G -cat $\rightsquigarrow \{\text{fin-clas}^2\text{-Grpd}\}$

- For \mathcal{G} a restricted Lie alg

$$\text{rep}^{\text{res}}(\mathcal{G}) = \left\{ \begin{array}{l} \text{fin dim restricted} \\ \text{rep's of } \mathcal{G} \end{array} \right\} \xrightarrow{(\rho)} G^V = (\mathcal{X} \rightarrow)^p G^V$$

- For G a finite group scheme

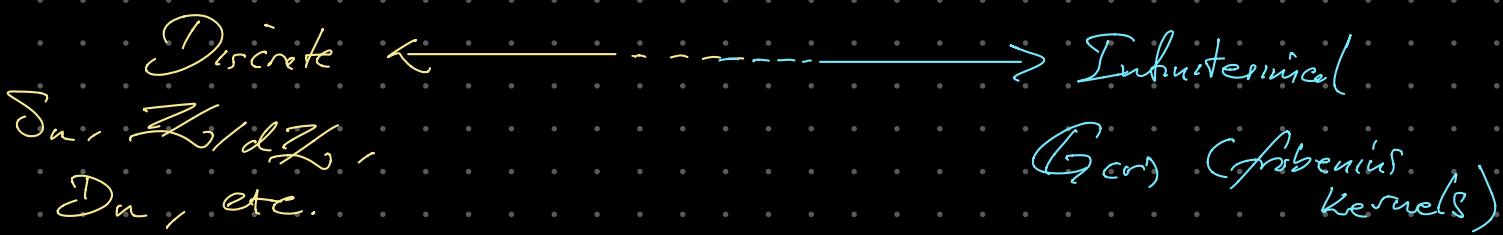
$$\text{Coh}(G) = \left\{ \begin{array}{l} \text{discrete} \\ \text{sheaves on } G \end{array} \right\} = \text{rep}(\mathcal{O}(G))$$

$$\stackrel{\cong}{\longrightarrow} \text{rep}(G)$$

- Reminders on finite group schemes

④

Recall that a finite group scheme G is the spectrum $G = \text{Spec}(G)$ of a fin-clas² comm Hopf alg \mathcal{G} .



Recall that for any affine group scheme G have p^n -th power map on the alg of funs. Dually, we have a map of schemes. For called the n -th Frobenius

$$1 \rightarrow G_{(n)} \rightarrow G \xrightarrow{F_n} G^{(n)} \rightarrow 1$$

This is a faithfully flat map of group schemes, and its kernel $G_{\text{cris}} := \ker(F_n)$ is a finite subgroup in G with a single point.

For G a finite group scheme we can consider alg of functions $\mathcal{O}(G)$ or the group ring $\kappa G := \mathcal{O}(G)^*$.

Have

$$\text{Coh}(G) = \text{rep}(\mathcal{O}(G))$$

$$\text{rep}(G) = \text{rep}(\kappa G).$$

- The center $Z(\text{rep } G)$

⑤

For C a finite \mathbb{D} -cat have fundamental construction, called the Drinfeld Center of C ,

$$Z(C) = \left\{ \begin{array}{l} \text{Cat of central} \\ \text{objects in } C \end{array} \right\} = \left\{ \begin{array}{l} \text{Cat of pairs } (V, \beta) \\ \text{w/ } V \text{ in } C \text{ and} \\ \text{a nat from } \Xi: V \otimes - \xrightarrow{\cong} \\ - \otimes V \end{array} \right\}$$

$Z(C)$ is a fin \mathbb{D} -cat which is "twice as big" as C , and it controls the monoidal theory of C . For $\text{rep}(G)$ have elegant interpretation.

$$Z(\text{rep } G) = \left\{ \begin{array}{l} \text{Cat of } G\text{-equiv} \\ \text{cols s.t. on } G \\ \text{under the adj} \\ \text{action } G \\ \text{adj } G \end{array} \right\} =: \text{Coh}(G)^G$$

$$= \text{rep}(\mathcal{O} \rtimes {}^G \text{ad } G)$$

called Drinfeld double. ↗

- Theorem

Thm [EN, EN18]: For G any finite group scheme, canon over $\mathbb{Z} = \mathbb{Z}(\text{rep } G)$ has the following strong finiteness prop's:

(FG1) For V any obj in \mathbb{Z} , the extensions $\text{Ext}_{\mathbb{Z}}(V, V)$ form a fin gen^l \mathbb{Z} -alg, which is fin over its center.

(FG2) For any other W in \mathbb{Z} , $\text{Ext}_{\mathbb{Z}}(V, W)$ is a fin gen^l module over ext^ls of V on the right and ext^rs of W on the left.

(FG3) For all V in \mathbb{Z}

Always a conn algbr!

$$\text{Kdim } \text{Ext}_{\mathbb{Z}}(V, V) \leq \text{Kdim } \text{Ext}_{\mathbb{Z}}(1, 1)$$

"Known" $\xrightarrow{\quad G \quad}$ *"Known"*

$$\approx \text{Kdim } \text{Ext}_G^*(1, 1) + \text{embd. dim}(G).$$

Thm [EN18]: For G a smooth alg group,
 $G = G^{(n)}$ on n -th Frobenius Kernel, and $\mathfrak{g} = \text{Lie}(G)$,
have

$$(*) \quad \text{Spec } \text{Ext}_{\mathbb{Z}}(1, 1) \approx \text{Spec } \text{Ext}_G^*(1, 1) \times \mathfrak{g}^{(n)}$$

Furthermore, $(*)$ is a homeomorphism when G is, for example,
when G is almost-simple / p and p is sufficiently large.

... But, where are these results coming from...

- Some context

(9)

Let's say finite \mathbb{Q} -cat C is of finite type [over the base field \mathbb{Q}] if its cohom satisfies the strong finiteness conditions (FG1)–(FG5).

Historically:

- [Evens, Golod, Venkov '05] For G a discrete finite group, the rep cat $\text{rep}(G)$ is of finite type.
- [Friedlander-Suslin '97] For G a fin. group scheme, $\text{rep}(G)$ of finite type. Also, [Fr-Pevsner, SF Bendel] can describe the spectrum of cohom as a certain moduli space of "special nilpotent elements" in $\mathbb{Q}G$. (a nil-cone).
- [Drapieski '10] $\text{rep}(SG)$ for a super group scheme ...

[Many examples in char 0, which I want recall]

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Have

Conj [N-Plavnik]: If C of finite type $\Rightarrow Z(C)$

of finite type. Furthermore, in char 0,

$$\text{Klein}(\text{Chom } Z(C)) = 2 \cdot \text{Klein}(\text{Chom } C).$$

Conj [Elmendorf-Goto '05] Aug finite \mathbb{Q} -category
is in fact of finite type.

Remark: The scheme $\text{Spec}(\text{Chom } C)$ should "determine the global structure" of the derived category. Furthermore, $\text{Spec}(\text{Chom } C)$ should have an interpretation in terms of a 4-dim field theory, dict by C (when C is ...).

- Methods (defn theory)

(11)

Take again $\mathbb{Z} = \mathbb{Z}(\text{rep } G)$. G a finite group scheme.
Let's just consider extensions of the unit \mathbb{I} .

Have

$$\text{rep } G \xrightarrow[\substack{\text{supp at} \\ \mathbb{I}}]{\text{shur}} \mathbb{Z} = \text{Coh}(G)^G$$

$$\Rightarrow \text{Ext}_G^i(\mathbb{I}, \mathbb{I}) \longrightarrow \text{Ext}_{\mathbb{Z}}^i(\mathbb{I}, \mathbb{I})$$

This gives a part of cohom, contributed from $\text{rep}(G)$.
We need a contribution from $\text{Coh}(G)$, or $\mathcal{O}(G)$, in
order to fully group cohomo for \mathbb{Z} .

We employ deformation theory for this.

Consider $G = \mathbb{G}_{\text{con}}$. Have

$$\begin{array}{ccc} \mathbb{G}_{\text{con}} & \longrightarrow & \mathbb{G} \\ & & \searrow F_{\text{irr}} \\ & & \mathbb{G}^{(\text{irr})} \end{array} \quad (*)$$

realizing \mathbb{G} as a flat family of schemes/synt over $\mathbb{G}^{(\text{irr})}$
which deforms \mathbb{G}_{con} .

$$\Rightarrow \mathbb{T}_1 \mathbb{G}^{(\text{irr})}$$

$$\mathbb{A}_{\text{defn}} := \text{Sym}(\mathbb{V}^{*(\text{irr})}) \longrightarrow \text{Ext}_{\text{Coh}(G)}^i(\mathbb{I}, \mathbb{I})$$

[Standard defns
fury, Gerstenhaber]

Cohom day 2.

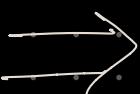
(12)

(18)

But this deformation is in fact equivariant in the sense that the adj action of $G^{(n)}$ on \mathcal{G} restricts to an action on the fibers over $G^{(n)}$

$$\begin{array}{ccccc} G^{(n)} & \longrightarrow & G & \xrightarrow{F_n} & G^{(n)} \\ \text{adj } \mathcal{E} & & \text{adj } \mathcal{E} & & \\ G^{(n)} & & G^{(n)} & & \end{array}$$

So \mathcal{G} deforms $G^{(n)}$ as a $G^{(n)}$ -scheme



$$A_{\text{defo}} = \text{Spec}(g^{*(n)}) \longrightarrow \text{Ext}_{\mathbb{Z}}^1(I, I).$$

Then $\{F_n\}$:

The product map $\text{Ext}_{\mathbb{Q}}^1(I, I) \otimes A_{\text{defo}} \xrightarrow{\text{finite}} \text{Ext}_{\mathbb{Z}}^1(I, I)$

The general case is not like this...

Can embed \mathcal{G} in coh. f.g. group scheme into smooth G and obtain a defo

$$\begin{array}{ccc} \mathcal{G} & \hookrightarrow & G \\ & & \searrow \\ & & \text{flat} \end{array} \longrightarrow \mathcal{G}/G$$

But the fibers in \mathcal{G} are permuted by the adj action of G .

This is because G not normal in G in general, and

so acts on parameter space G/G .

But, can consider a new kind of "group deformations"

Here we allow G to act on the param space, and look for higher deformation classes

$$\text{Defo}^{\text{higher}} \rightarrow \text{Ext}_{\mathbb{Z}}(I, I)$$

Same $\text{Ext}_G(I, I)$ with generators in high degree

[in N]: We can in fact do this, and the moral map

$$\text{Ext}_G(I, I) \otimes \text{Defo}^{\text{higher}} \rightarrow \text{Ext}_{\mathbb{Z}}(I, I)$$

(or again fourth)

