POSTNIKOV DIAGRAMS AND ORBIFOLDS

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① Surface combinatorics ⇝ cluster structures

triangulations of		cluster algebra/category of	
convex n-gon	⇝	type A_{n-3}	Fomin-Zelevinsky 02, Caldero-Chapoton-Schiffler 06
convex n-gon w. puncture	⇝	type D_n	FZ02, Schiffler 08
marked surface	⇝		Fomin-Shapiro-Thurston, Brüstle-Zhang
annuli	⇝	$\tilde{A}$	FST, B-Marsh, B-Buan-Marsh

replace triang. (surface = disk, mostly)

Idea: triangulation of surface gives "cluster" (collection of variables / of indec. obj) and mutation rule

Postnikov diagram

alternating strand diagram of type (k,n) ⇝ cluster structure for Grassmannian — Scott 06, B-King-Marsh 16

Postnikov diagrams only give some clusters

Today: introduce orbifold diagrams as quotients of symmetric Postnikov diagrams

1

$1<k<n$

$Gr(k,n) = \{k\text{-dimensional subspaces of } \mathbb{C}^n\}$ proj. variety [think of $k\times n$-matrices$/GL_k$ or $\Lambda^k\mathbb{C}^n$]

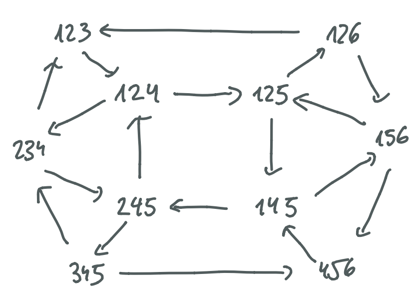
embedded in $\mathbb{P}^N$; Plücker rel's

Theorem (Scott '06) The coordinate ring $\mathbb{C}[\widehat{Gr(k,n)}]$ has a cluster algebra structure: ∃ seeds of Plücker coordinates;

all Plücker coord's are cluster variables

mutation from Plücker relations

eg (3,6)



eg $\{124, 125, 245, 145\} \cup \{i,i+1,i+2 \mid i=1,\dots,6\}$ with mutation rule encoded in quiver

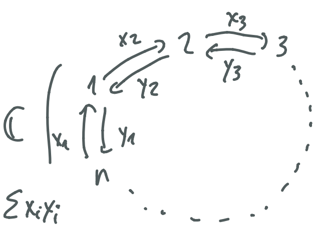
mutable, frozen, reduce mod 6

$p_{124} \sim e_1\wedge e_2\wedge e_4$

2

Cluster category for Grassmannian

Jensen-King-Su '16

$B_{k,n} := \mathbb{C}$  $/ \left\{ \begin{matrix} xy - yx \\ x^k - y^{n-k} \end{matrix} \right\}$

take completed path alg.

add. cat. of above cl. alg. str.: $Z = \mathbb{C}[[t]]$, $t = \sum x_i y_i$

$\mathcal{F}_{k,n} := \{ \text{m CM for } B_{k,n} \} = \{ M \mid M \text{ free over } Z \}$

- ∞ dim mod's
- of inf. rep. type in general

rank 1-modules are in bijection with k-subset (hence w. Plücker coord's)

ext-orthogonal $\Leftrightarrow$ non-crossing

B-King-Marsh '16 $\quad \mathcal{F}_{k,n}$ has cluster-tilting objects given by max^l non-crossing collections of k-subsets (with proj.-inj. summands corr. to the frozen variables $p_{i,i+1,\dots,i+k-1}$)

for any such $T \in \mathcal{F}_{k,n}$: $B_{k,n}^{op} \cong e(\operatorname{End} T)e$

e idempotent of boundary vertices

$\boxed{3}$

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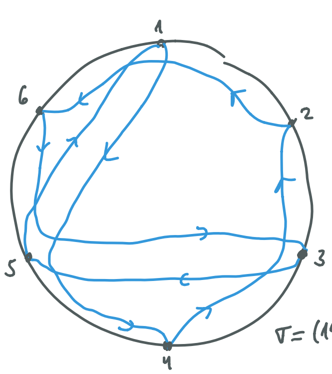
2 Postnikov diagrams

D_n disk with n points on boundary

$\sigma \in S_n$ permutation

A Postnikov diagram P of type σ on D_n is a collection of n strands (oriented curves) $\gamma_1, \dots, \gamma_n$, $\gamma_i : i \longmapsto \sigma(i)$ (up to isot. fixing bdy)

- $*_1$ transversal crossings, mult. 2, fin. many
- $*_2$ crossings alternate
- $*_3$ no unoriented lenses
- $*_4$ any loop defines a disk with no other strands inside (except at bdy)



$\sigma = (142635)$

P is of type (k,n) if σ is $i \mapsto i+k$

P is d-symmetric if it is invariant under rotation by $2\pi/d$ $/\sim$ isotopy

Pasquali (self-inj. alg's)

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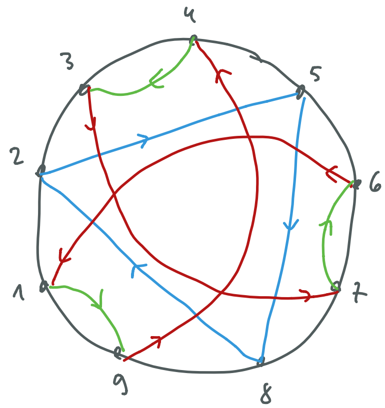
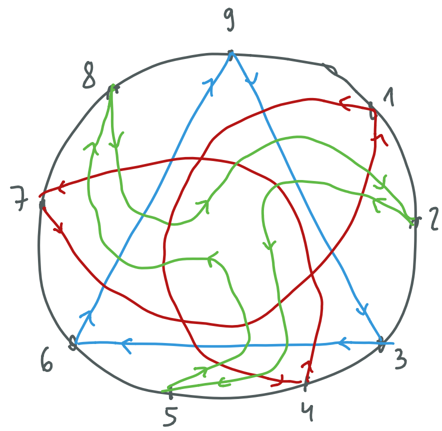
Reductions:

Examples of symmetric Postnikov diagrams



d=3

$\sigma = (194376)(258)$



d=3

of type (3, 9)

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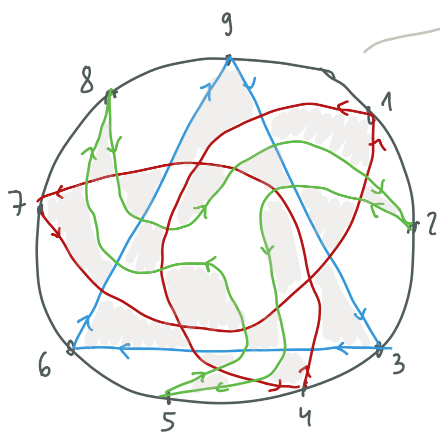
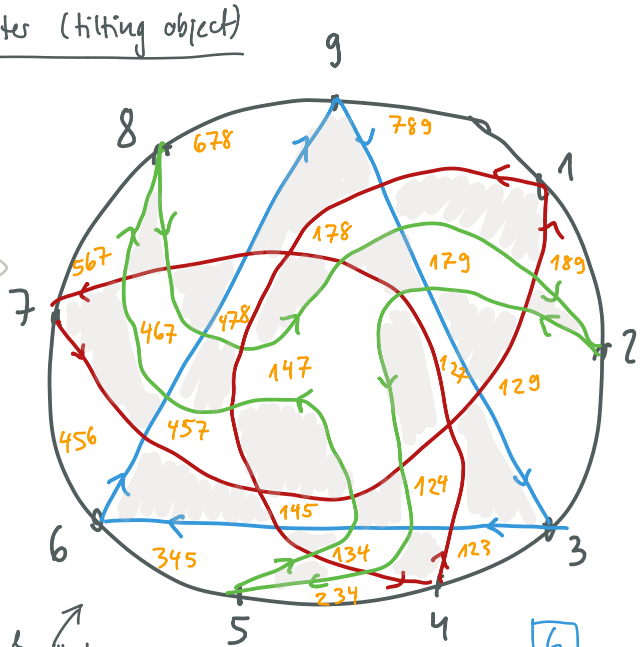
3. Postnikov diagrams ⇝ cluster (tilting object)

Label alternating regions with i if to the left of σ_i





max^l collection of noncrossing 3-subsets of 9

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4. Orbifold diagrams

Idea: take quotient by d-rotational symm.

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Define it abstractly first

$\Sigma = \Sigma_{n_0}$ disk with points $1, \dots, n_0$ and orbifold pt Ω of order $d > 1$

A <u>weak orbifold diagram</u> of type $\tau \in S_{n_0}$ on Σ is a coll. of n_0 strands (oriented curves) $c_i : i \mapsto \tau(i)$ s.t.
- * c_i do not go through Ω
- * fin. many crossings, transverse, mult. 2
- * crossings alternate
- * non-oriented lenses contain Ω
- * any loop formed by a strand has nonzero winding # around Ω (if it can't be reduced)

up to isotopy fixing boundary & Ω

[reduce like P. diagrams]

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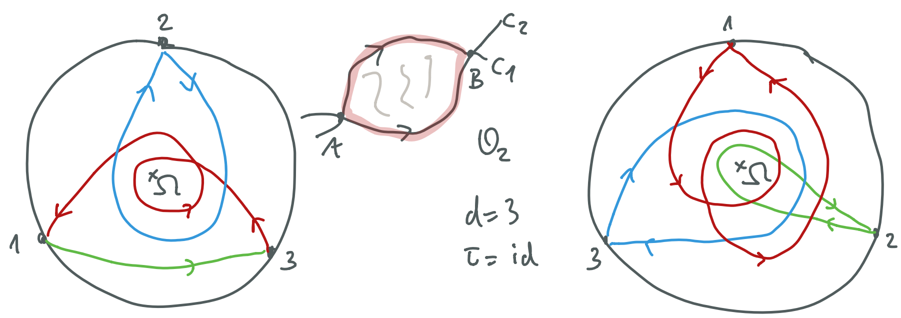
<u>Examples</u>

$\mathcal{O}_1$ $d=3$ $\tau = (13)$

$\mathcal{O}_2$ $d=3$ $\tau = id$



$\mathcal{O}$ weak orbifold diagram, $ord(\Omega) = d$. Let $sym_d(\mathcal{O})$ be the d-fold cover

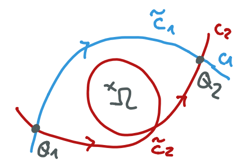
Want $sym_d(\mathcal{O})$ to be a Postnikov diagram. Problematic *₃ no unoriented lenses
*₄ no non-trivial loops

For *₄

<u>Winding value</u> of strand $c \in \mathcal{O}$: $S(c) := \max_{P \subset c \cap c} |w(P)|$ ↖ winding # w.r.to Ω

For *₃:

$c_1 \neq c_2$ strands of $\mathcal{O}$ $L(c_1, c_2) := \max_{c_1 \neq c_2} |w(\tilde{c}_1 \tilde{c}_2^{-1})| \geq 0$ ↑ winding # of curve $\tilde{c}_1 \tilde{c}_2^{-1}$



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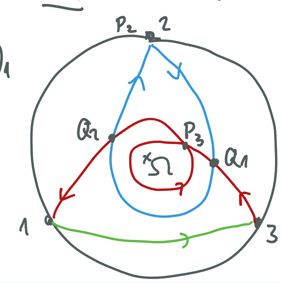
A weak orbifold diagr. $\mathcal{O}$ on Σ with Ω of order $d > 1$ is an orbifold diagram if $d > \max\{\max S(c), \max L(c_1, c_2)\}$

$c_1 \neq c_2$

It is <u>Grassmannian</u> if $\tau = id$ and $\exists\ 0 < w_+ < d$ s.t. every strand has winding # w_+ or $w_+ - d$.

<u>Ex.</u>: $\mathcal{O}_1$ and $\mathcal{O}_2$ are orbifold diagrams, $\mathcal{O}_2$ is Grassmannian.

$\mathcal{O}_1$



$\left.\begin{array}{l} w(P_2) = 1 \\ w(P_3) = -1 \end{array}\right\}\ S(c_2) = S(c_3) = 1$

$w(Q_1 Q_2) = 2 \Rightarrow L(c_2, c_3) = 2$

$d > \max\{1, 2\}$ ✓

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<u>Proposition</u> $\mathcal{O}$ orbifold diagr. of order $d > 2$, $\mathcal{P}$ an s-symm. Postnikov diagram $(s > 1)$

(1) $sym_d(\mathcal{O})$ is a Postnikov diagram $\boxed{\mathcal{P}/s: \text{ quotient by } 1/s\text{-rotation}}$

(2) $\mathcal{P}/s$ is an orbifold diagr. on disk w. order s orbifold pt

(3) $sym_d(\mathcal{O})/d = \mathcal{O}$ and for $s > 2$, $sym_s(\mathcal{P}/s) = \mathcal{P}$.

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5. Labels for orbifold diagrams

Prop: $sym_d(\mathcal{O})/d = \mathcal{O} \rightsquigarrow$ labels as equiv. classes

$\mathcal{O}$ of order $d > 2$, of type $\tau \in S_{n_0} \rightsquigarrow sym_d(\mathcal{O})$ on disk with $n := d n_0$ pts.

$\mathcal{I}$ Let $\mathcal{I}$ be the set of labels of $\mathrm{sym}_d(\mathcal{O})$. Define equiv. relation $\sim_{n_0}$:
$\{i_1,\dots,i_k\} \sim_{n_0} \{h_1,\dots,h_k\}$ if there exists j s.t. $\{i_1+jn_0,\dots,i_k+jn_0\} = \{h_1,\dots,h_k\}$ [reduce mod n]

[Labels of two regions related by rotation by $2\pi/d$ differ by adding n_0 pointwise]

Every alternating region of $\mathcal{O}$ corr. to d (or 1) alt. region of $\mathrm{sym}_d(\mathcal{O})$, i.e. to an equiv. class under $\sim_{n_0}$. We write $[i_1,\dots,i_k]_{n_0}$.

Can assign labels directly to $\mathcal{O}$

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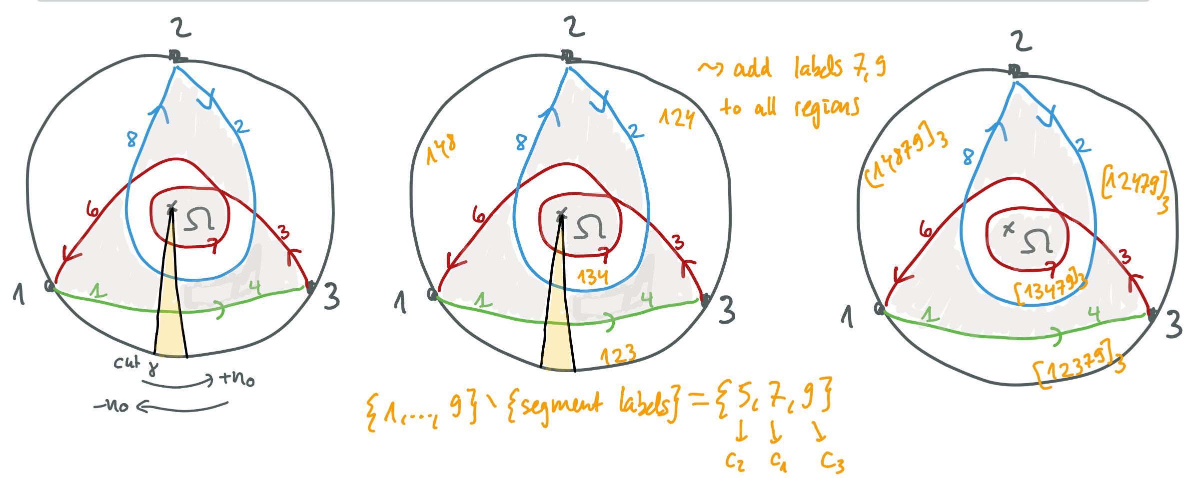

- Draw cut γ from Ω to boundary between n_0 and 1
- label regions to left
- label segments of c_i by $i, i+n_0, i+2n_0, \dots$ for c_i Ω and by $i, i-n_0, \dots$ if c_i Ω
- altern. region to left of segment label gets this label

regions cut in 2: only label part anticlockwise from γ

- For any $j \in \{1,\dots,n\} \setminus \{\text{segment labels}\}$: Let j_0 be the reduction of j mod n_0

If Ω c_{j_0} add label j to every alt. region of $\mathcal{O}$.

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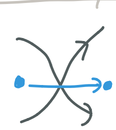
6. Quivers with potential

$\mathcal{O}$ orbifold diagram of order d

$Q_{\mathcal{O}}$: vertices the alternating regions of $\mathcal{O}$; If Ω is in an alternating region: $v_1,\dots,v_d$ in this region

(also boundary regions)

view the v_i as on vertical line above Ω

arrows from 

if $\exists\, v_1,\dots,v_d$: same for each of them.

no arrows between

[$Q_{\mathcal{O}}$ ~ dimer model]

[the v_i] with boundary, BKM16]
(up to $v_1,\dots,v_d$)

For potential: $\mathcal{C} = \{$fundamental cycles in $Q_{\mathcal{O}}\}$

(a) Ω in fund. cycle $\Rightarrow$ c the fund. cycle containing Ω

(b) Ω not in fund. cycle $\Rightarrow$ Ω is adj. to r fund. cycles $\Rightarrow$ $c_i^{(1)},\dots,c_i^{(r)}$ the fund. cycles through v_i

If $\exists$ $v_1,\dots,v_d$: $Q_{\mathcal{O}}$ from quiver w. faces on annulus. Glue d isom. disks on inner bdy of annulus

$W_{\mathcal{O}}$ potential on $Q_{\mathcal{O}}$: (a) $W_{\mathcal{O}} := \sum_{c' \in \mathcal{C}\setminus\{c\}} \operatorname{sgn}(c')c' + \frac{1}{d}\operatorname{sgn}(c)c^d$

(b) $W_{\mathcal{O}} := \sum_{c' \in \mathcal{C}\setminus\{c\}} \operatorname{sgn}(c')c' + \sum_{j\neq r}\sum_{i=1}^{d} \operatorname{sgn}(c_i^{(j)})c_i^{(j)} + \sum_{i=1}^{d} \zeta^i \operatorname{sgn}(c_i^{(r)})c_i^{(r)}$

ζ primitive d-th root of 1

as potential in Giovannini-Pasquali
GP Plamondon

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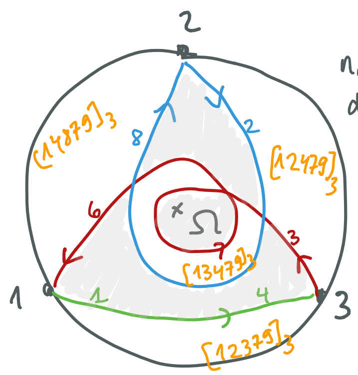
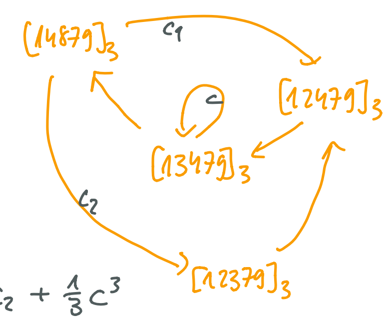
Ex.

$\mathcal{O}_1$



$n_0 = 3$
$d = 3$

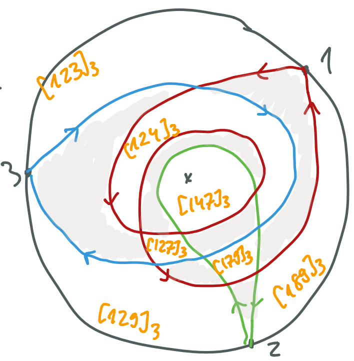
$Q_{\mathcal{O}_1}$



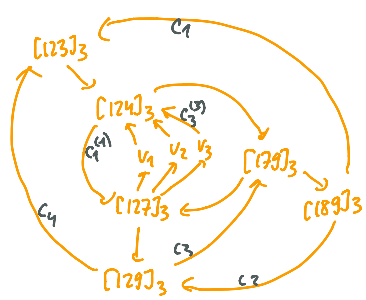
$W_{\mathcal{O}_1}$:

$-c_1 + c_2 + \frac{1}{3}c^3$

$\mathcal{O}_2$



$n_0 = 3$
$d = 3$



$Q_{\mathcal{O}_2}$

$W_{\mathcal{O}_2}$: $c_1 - c_2 + c_3 - c_4 + c_1^{(1)} + c_2^{(1)} + c_3^{(1)} - c_1^{(2)} - c_2^{(2)} - c_3^{(2)}$

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7. Orbifold and boundary algebras

$\mathcal{O}$ orbifold diagram of order d

[assume reduced]

as BKM16: $\mathcal{P} = \operatorname{sym}_d(\mathcal{O})$ Postnikov diagram

[bdy vertices frozen]

$A(\mathcal{P})$ (completed) frozen Jacobian alg. of $(Q_{\mathcal{P}}, W_{\mathcal{P}})$

$B(\mathcal{P}) := eA(\mathcal{P})e$ its boundary alg. (e idemp. of the bdy vertices)

$\rightsquigarrow$ $A(\mathcal{O}) :=$ frozen Jacobian algebra of $(Q_{\mathcal{O}}, W_{\mathcal{O}})$

and $B(\mathcal{O}) := eA(\mathcal{O})e$ e the idempotent of the boundary vertices

Skew group algebra: S algebra (work over $\mathbb{C}$) w. group action by G (finite; for us: cyclic of order d)

The <u>skew group algebra</u> $S * G$ is $S \otimes_{\mathbb{C}} \mathbb{C}G$,

multipl. induced from $(s \otimes g)(t \otimes h) = s\,g(t) \otimes gh$ $\quad s,t \in S$, $g,h \in G$

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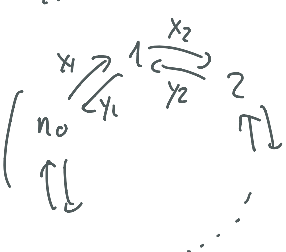
G gen. by rotation by $2\pi/d$ acts on algebras $A(\mathcal{O})$, $B(\mathcal{O})$

$A(\mathcal{O}) \sim A(\mathcal{P}) * G \qquad B(\mathcal{O}) \sim B(\mathcal{P}) * G$

Assume $\mathcal{O}, \mathcal{P}$ are Grassmannian of type (k,n)

$\mathcal{O}$ on disk w. n_0 vertices

$\mathcal{P}$ — " — $n = n_0 d$ vertices

$B_G := \mathbb{C}\Big($  $\Big) \Big/ \left\{ \begin{array}{l} xy - yx \\ x^k - y^{n-k} \end{array} \right\}$ (completed path alg.)

<u>Theorem</u> (1) $B(\mathcal{O}) \cong (B_G)^{op}$

(2) $A(\mathcal{O}) \cong \mathrm{End}_{B_G}(T_{\mathcal{O}})$ $\qquad T_{\mathcal{O}} := \bigoplus_{[I] \in \mathcal{I}_{\mathcal{O}}} L_{[I]}$ (analogues to the rk 1 modules in $\mathcal{F}_{k,n}$)

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