Simple-mindedness: negativity and positivity

Joint with Raquel Coelho Simões and David Ploog.

__Aim:__ Convince you simple-minded systems are "cluster-tilting objects".

__Recall__ D Hom-finite, k-linear, Krull-Schmidt triangulated category with shift $\Sigma : D \to D$. A __Serre functor__ on D is an autoequivalence $\mathbb{S}: D \to D$ s.t.

$$\mathrm{Hom}(x, y) \simeq D\,\mathrm{Hom}(y, \mathbb{S}x) \quad \text{for } x, y \text{ objects of } D.$$

For $w \in \mathbb{Z}$, D is __w-Calabi-Yau__ (w-CY) if $\Sigma^w \simeq \mathbb{S}$.

__Theorem__ (Reiten-van den Bergh)

D has a Serre functor $\iff$ D has AR triangles, in which case $\tau = \Sigma^{-1}\mathbb{S}$.

__Protagonist:__ Q (finite) acyclic quiver, $w \in \mathbb{Z} \setminus \{0, 1\}$. Set

$$C_w = D^b(kQ) / \Sigma^{-w}\mathbb{S}, \quad \text{this is } w\text{-CY}.$$

$w \geq 2$, C_w is a __(higher, classical) cluster category__.

$w \leq -1$, C_w is a "__classical negative cluster category__"

Objects of C_w = objects of $D^b(kQ)$

Morphisms of C_w: $\mathrm{Hom}_{C_w}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b(kQ)}(X, F^i Y)$, $F = \Sigma^{-w}\mathbb{S}$.

$w \leq -1$ recovers natural constructions:

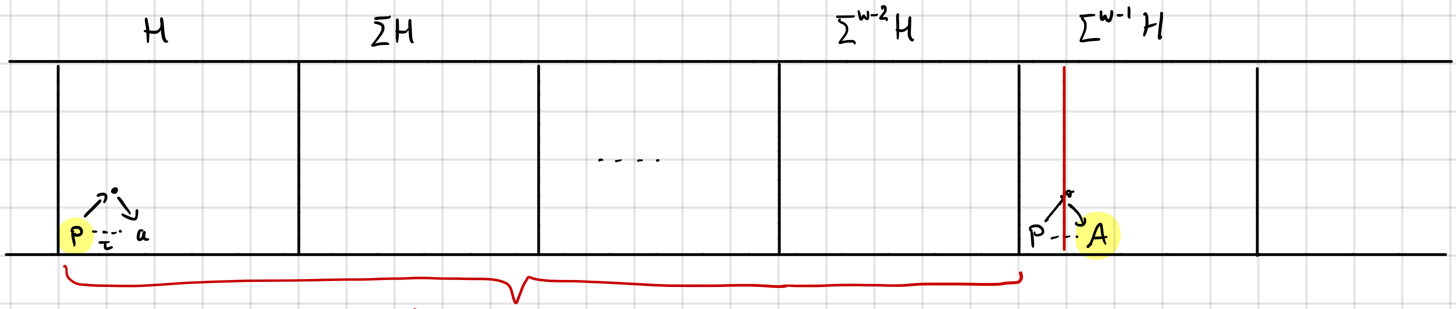
* $w = -1$, stable module categories of rep^n-finite symmetric algebras (Coelho Simões, Riedtmann)
* CM A, w-selfinjective dg algebras (Brightbill, Jin).

__Theorem__ (Keller, 2005)

The projection functor $\pi : D^b(kQ) \longrightarrow D^b(kQ)/\Sigma^{-w}\mathbb{S} = C_w$ is a triangle functor, which gives rise to an additive equivalence $\pi : \mathcal{F}_w \longrightarrow C_w$.

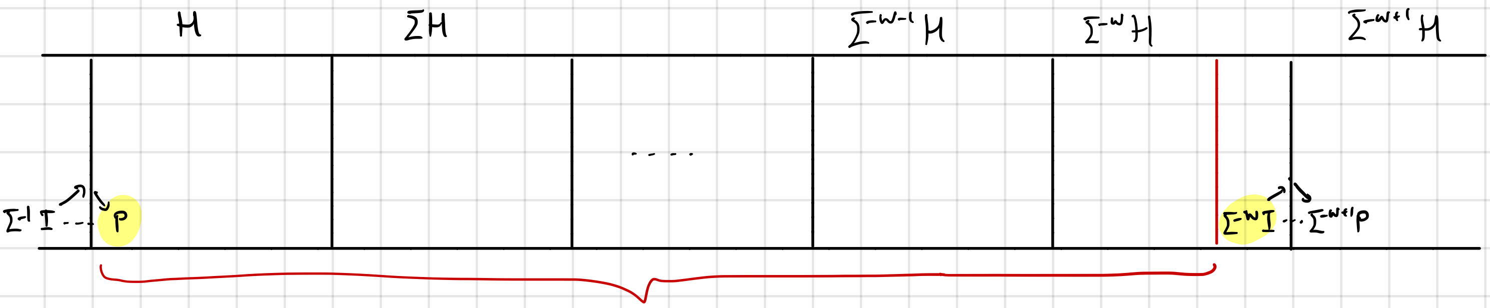
Schematics

1) $w \geq 2$. Write $\mathcal{H} = \operatorname{mod} kQ$.



$\mathcal{F}_w$ = fundamental domain = $\mathcal{H} \cup \Sigma\mathcal{H} \cup \cdots \cup \Sigma^{w-2}\mathcal{H} \cup \Sigma^{w-1} \operatorname{proj} kQ$

2) $w \leq -1$.



$\mathcal{F}_w$ = fundamental domain = $\mathcal{H} \cup \Sigma\mathcal{H} \cup \cdots \cup \Sigma^{-w}\mathcal{H} \setminus \Sigma^{-w}(\operatorname{inj} kQ)$.

Projective-minded versus simple-minded

Theorem (König–Yang, 2014)

Let Λ be a finite dimensional algebra. There is a bijection

$$\{\text{algebraic t-structures in } D^b(\Lambda)\} \overset{1-1}{\longleftrightarrow} \{\text{SMCs in } D^b(\Lambda)\} \overset{1-1}{\longleftrightarrow} \{\text{silting objects in } K^b(\operatorname{proj}\Lambda)\}/\sim$$

$$\operatorname{mod}\operatorname{End}_{D^b(\Lambda)}(M) \simeq \langle S \rangle \longleftarrow S \longmapsto M$$

Precurser: Let Q be a (finite) acyclic quiver, $\mathcal{P} = \operatorname{proj} kQ$, $\mathcal{H} = \operatorname{mod} kQ$.

For a silting object M, write $\mathcal{M} = \operatorname{add} M$ for the silting subcategory.

Theorem (Buan–Reiten–Thomas, 2012)

For $w \geq 2$, there are bijections, where W_Q is the corresponding Weyl group,

$$\left\{\begin{array}{l}\text{simple-minded collections of } D^b(kQ) \\ \text{lying in } \mathcal{H} \cup \cdots \cup \Sigma^{w-1}\mathcal{H}\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{silting subcategories } \mathcal{M} \text{ with} \\ \mathcal{M} \subseteq \mathcal{P} * \Sigma\mathcal{P} * \cdots * \Sigma^{w-1}\mathcal{P} = \mathcal{F}_w\end{array}\right\}$$

$$\updownarrow 1-1 \qquad\qquad\qquad\qquad \updownarrow$$

$$\{w\text{-noncrossing partitions of } W_Q\} \overset{1-1}{\longleftrightarrow} \{w\text{-cluster tilting objects of } \mathcal{C}_w\}/\sim$$

Theorem (Coelho Simões – P – Ploog, 2020)

Let $w \geq 1$, Q (finite) acyclic quiver, W_Q corresponding Weyl group. $\exists$ bijections

$$\begin{tikzcd}[column sep=tiny]
 & \{\text{simple-minded collections } S \text{ of } D^b(kQ) \text{ with } S \subseteq \mathcal{F}_{-w}\} & \\
\{\text{positive } w\text{-noncrossing partitions of } W_Q\} \arrow[ur, leftrightarrow, "\text{Ⓐ}", "1\text{-}1"'] \arrow[rr, leftrightarrow, "\text{Ⓒ}", "1\text{-}1"'] & & \{w\text{-simple minded systems of } \mathcal{C}_{-w}\} \arrow[ul, leftrightarrow, "\text{Ⓑ}"', "1\text{-}1"]
\end{tikzcd}$$

Remarks

Ⓒ was known for Q Dynkin and $w = 1$ (Coelho Simões, 2012)

Ⓐ was known for Q Dynkin and $w \geq 1$ (Buan – Reiten – Thomas, 2012)

Ⓑ was known for Q Dynkin and $w \geq 1$ (Iyama – Jin, 2020).

Simple-minded systems/collections

Definitions

A collection of objects $S \subseteq D$ is an <u>orthogonal collection</u> if $\operatorname{Hom}(x, y) = \delta_{xy} \cdot k \quad \forall x, y \in S$.

Let $w \geq 1$. An orthogonal collection is called

i) <u>w-orthogonal</u> if $\operatorname{Hom}(\Sigma^k x, y) = 0$ for $1 \leq k \leq w-1$, $x, y \in S$;

ii) <u>w-SMS</u> if it is w-orthogonal and $D = \langle S \rangle * \Sigma^{-1}\langle S \rangle * \cdots * \Sigma^{1-w}\langle S \rangle$;

iii) <u>w-Riedtmann</u> if it is w-orthogonal, $\bigcap_{k=0}^{w-1} (\Sigma^k S)^{\perp} = 0$ and $\bigcap_{k=0}^{w-1} {}^{\perp}(\Sigma^k S) = 0$;

iv) <u>∞-orthogonal</u> if $\operatorname{Hom}(\Sigma^k x, y) = 0$ for $k \geq 1$, $x, y \in S$; and

v) <u>SMC</u> if it is ∞-orthogonal and $D = \operatorname{thick}(S)$ ($\Leftrightarrow \langle S \rangle_D$ is heart of bdd t-str.)

Let $X \subseteq D$. A morphism $f: x \to d$ is a <u>right X-approximation</u> if

whenever we have

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, dashed, "\exists h"'] & d \\
x' \arrow[ur, "g"'] \arrow[u, phantom, shift left=3ex, "\circlearrowleft"] &
\end{tikzcd}$$

$X \ni x'$

If every object of D admits a right X-approximation then X is <u>contravariantly finite</u>.

Dually: left X-approximation, covariantly finite. <u>Functorially finite</u> = covariantly + contravariantly finite.

Proposition (Coelho Simões–P, 2020)

$S \subseteq D$ collection of indecomposable objects. Then S is a w-SMS in D iff S is a w-Riedtmann configuration in D s.t. $\langle S \rangle$ is functorially finite in D.

Slogan w-SMS = (higher) cluster-tilting subcategory

w-Riedtmann config = weakly (higher) cluster-tilting subcategory.

Theorem (Dugas)

If $S \subseteq T$, T orthogonal collection, then $\langle S \rangle$ is functorially finite in $\langle T \rangle$.

Brief aside – functorially finite hearts

Theorem (Coelho Simões–P–Ploog, 2020)

Let $\mathcal{H}$ be the heart of a bounded t-structure in D. The heart $\mathcal{H}$ is functorially finite in D iff $\mathcal{H}$ has enough injectives and enough projectives.

Upshot

If S is an SMC in $D^b(kQ)$ then $\langle S \rangle$ is functorially finite in $D^b(kQ)$.

Proof of (B)

Proposition (Iyama–Jin)

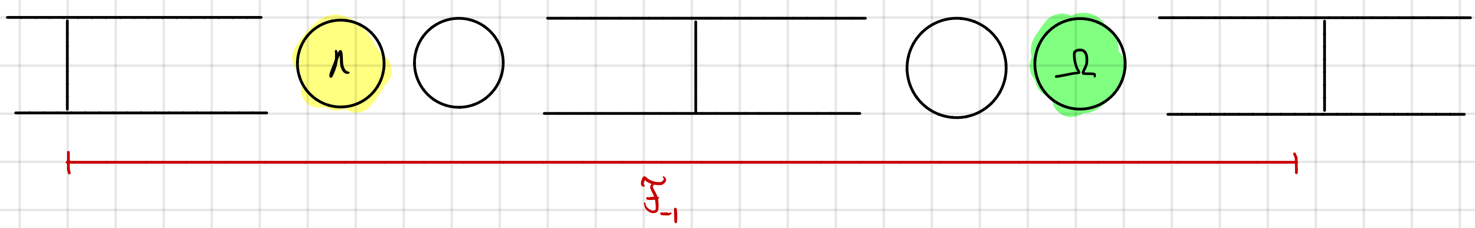
The natural projection functor $\pi : D^b(kQ) \longrightarrow \mathcal{C}_{-w}$ induces a well defined map

$$\{\text{SMCs of } D^b(kQ) \text{ contained in } \mathcal{F}_{-w}\} \hookrightarrow \{w\text{-Riedtmann configurations in } \mathcal{C}_{-w}\}.$$

This is bijective when Q is Dynkin.

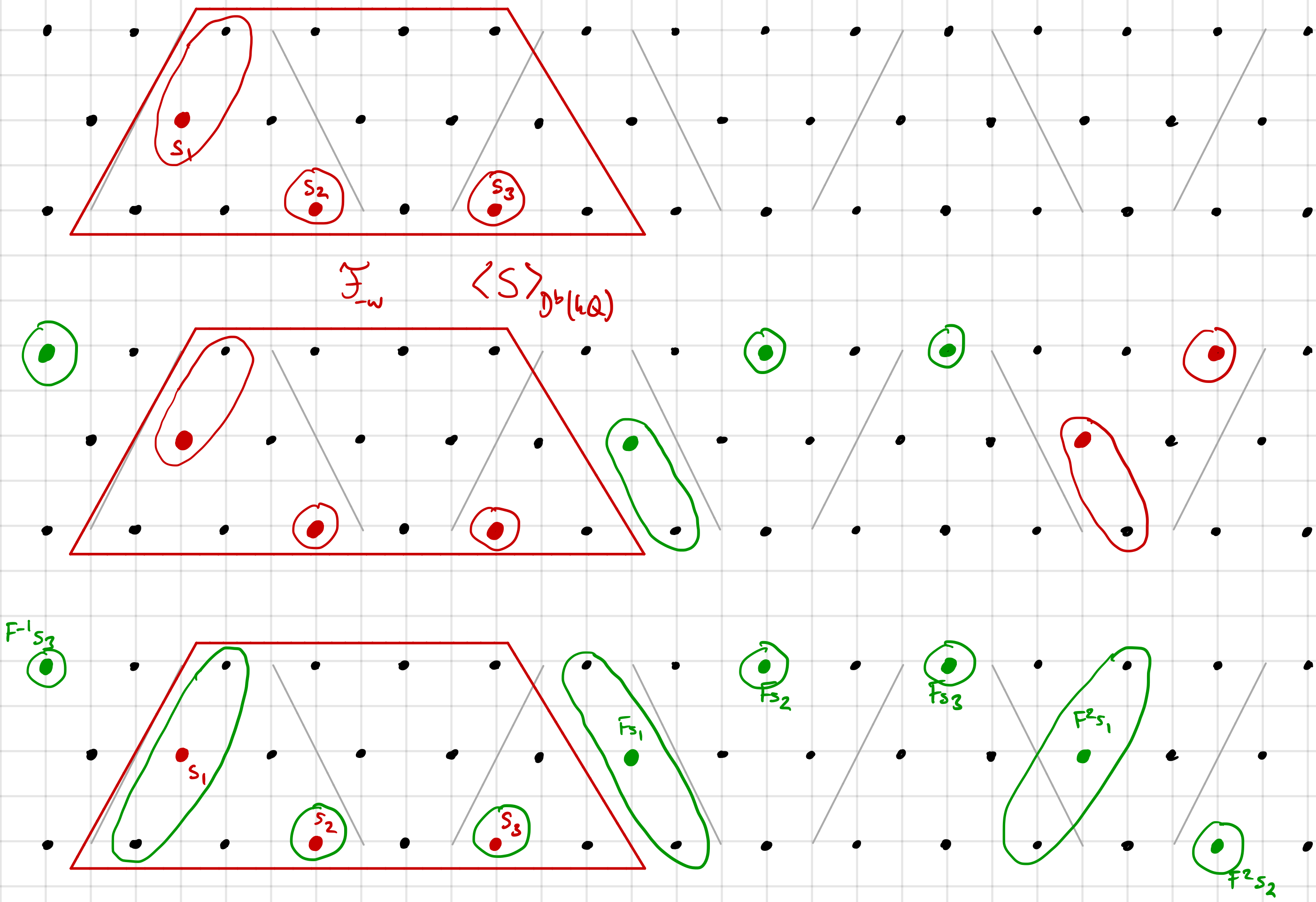
Problem

$Q = 1 \rightrightarrows 2$, $w = 1$. $S = \{S_\lambda \mid \lambda \in \Lambda\} \cup \{\Sigma S_\omega \mid \omega \in \Omega\}$ is a 1-Riedtmann config.



<u>Idea</u>: Use functorial finiteness of $\langle S\rangle_{D^b(kQ)}$ in $D^b(kQ)$ to show functorial finiteness of $\langle S\rangle_{C_{-w}}$ in C_{-w}.

<u>Example</u> $\quad C_{-2}(A_3) = D^b(kA_3)/_{\Sigma^{-3}\mathbb{S}} =: F$



Let $\mathcal{E}_S = \langle F^i S \mid i \in \mathbb{Z}\rangle_{D^b(kQ)}$. Then $\pi(\mathcal{E}_S) = \langle S\rangle_{C_{-w}}$.

In particular, $\mathcal{E}_S$ is functorially finite in $D^b(kQ)$ iff $\langle S\rangle_{C_{-w}}$ is functorially finite in C_{-w}.

But, $\langle S\rangle_D$ is functorially finite in $D^b(kQ)$ iff $\langle F^i S\rangle_{D^b(kQ)}$ is, and

$$\mathcal{E}_S = \cdots * \underbrace{\langle F^m S\rangle_D * \langle F^{m+1}S\rangle_D * \cdots * \langle F^n S\rangle_D}_{\mathcal{E}_S^{[m,n]}} * \cdots$$

[Saorín-Zvonareva] $\Rightarrow$ $\mathcal{E}_S^{[m,n]}$ is functorially finite

Hereditary property $\Rightarrow$ $\mathcal{E}_S$ functorially finite $\Leftrightarrow$ $\langle S\rangle_{C_W}$ functorially finite.

Surjectivity

Take S a w-SMS in $\mathcal{C}_{-w}$. [Iyama-Jin] show its lift to $D^b(kQ)$ is ∞-orthogonal collection.

$$\mathcal{E}_S = \overbrace{\cdots * \langle F^{-2}S\rangle_D * \langle F^{-1}S\rangle_D * \langle S\rangle_D}^{\mathcal{E}_S^{\leq 0}} \underbrace{ * \langle FS\rangle_D * \langle F^2 S\rangle_D * \cdots}_{\mathcal{E}_S^{\geq 0}}$$

- $\langle S\rangle_D$ is contravariantly finite in $\mathcal{E}_S^{\leq 0}$ and covariantly finite in $\mathcal{E}_S^{\geq 0}$.
- $\mathcal{E}_S^{\leq 0}$ is contravariantly finite in $\mathcal{E}_S$ by Dugas Theorem.

$\Rightarrow \langle S\rangle_D$ functorially finite in D $\Rightarrow$ $({}^{\perp}(\Sigma^{<0}S), \operatorname{cosusp} S)$ is a t-structure.

By hereditary property, show it's bounded.

Noncrossing partitions

Q quiver, $S_1, \dots, S_n$ simple kQ-modules $\rightsquigarrow$ $W_Q = \langle t_{S_1}, \dots, t_{S_n}\rangle$ corresponding <u>Weyl group</u>.

Call $t_{S_1}, \dots, t_{S_n}$ the <u>simple reflections</u>

Fix a <u>Coxeter element</u> c, i.e. a product of all simple reflections in some order, corresponding to an ordering of $S_1, \dots, S_n$ as an exceptional sequence.

A <u>parabolic subgroup</u> of W_Q is a subgroup generated a proper subset of R.

Definitions (Armstrong)

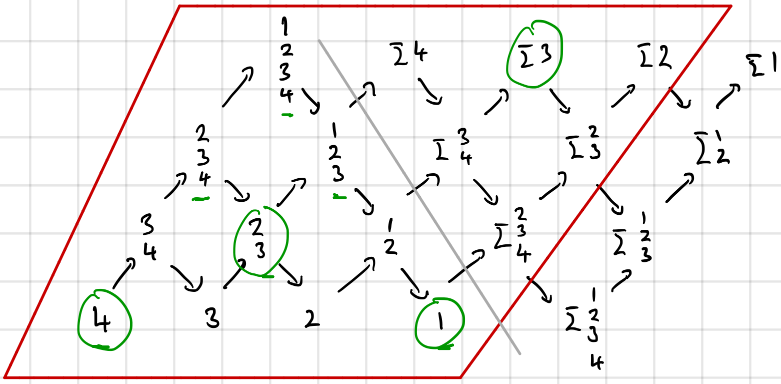
Let $w \geq 1$.
- $NC^w(W_Q) = \{\underline{u} = (u_1, \dots, u_{w+1}) \mid c = u_1 \cdots u_{w+1},\ l(u_1) + \cdots + l(u_{w+1}) = l(c)\}$.
- $NC^w_+(W_Q) = \{\underline{u} = (u_1, u_2, \dots, u_{w+1}) \mid \underline{u} \in NC^w(W_Q)$ and $u_2 \cdots u_{w+1}$ does not lie in any proper parabolic subgroup$\}$

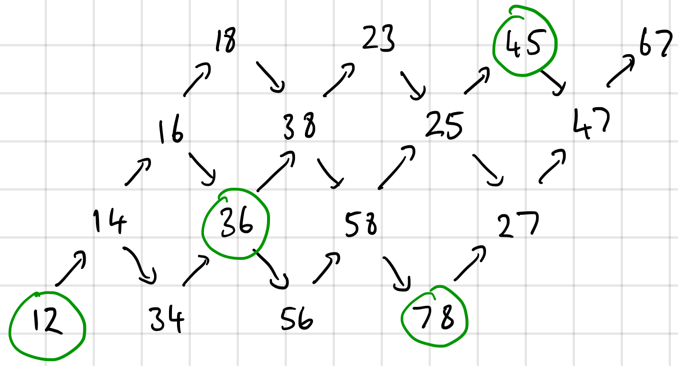
Example 1

$w = 1$, $Q = A_4 \ (1 \to 2 \to 3 \to 4)$, $W_Q \cong S_5$, $R = \{(12), (23), (34), (45)\}$

$c = (12)(23)(34)(45) = (1\ 2\ 3\ 4\ 5)$

Consider $c = \underbrace{t_3}_{u_1} \underbrace{t_{123}\, t_1\, t_4}_{u_2} = \underbrace{(34)}_{u_1}\underbrace{(14)(12)(45)}_{u_2}$ $\quad u_2 = (1\ 2\ 4\ 5)$



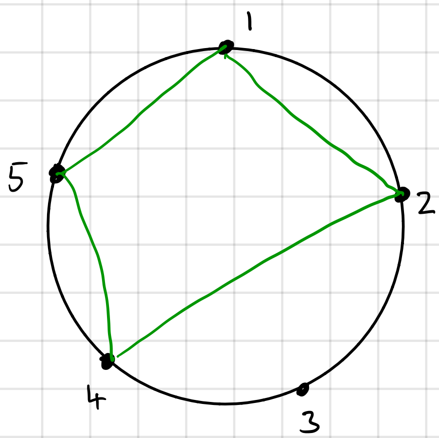


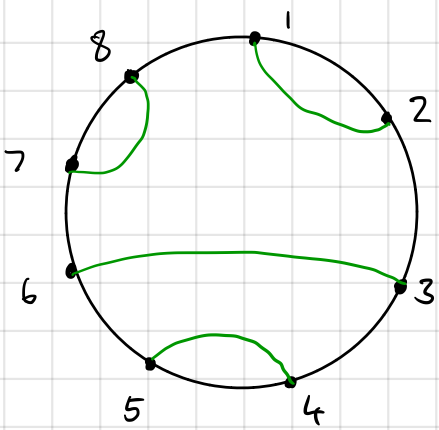
Noncrossing (Armstrong)

$NC(A_n) \simeq NC(n+1)$ = noncrossing partitions of $n+1$

$NC_+(A_n) \simeq NC_+(n+1)$ = noncrossing partitions of $n+1$ in which 1 and $n+1$ are in the same block

$\simeq NC^{pairs}(2n)$ = partition of $2n$ with noncrossing blocks and each block has only 2 elements.



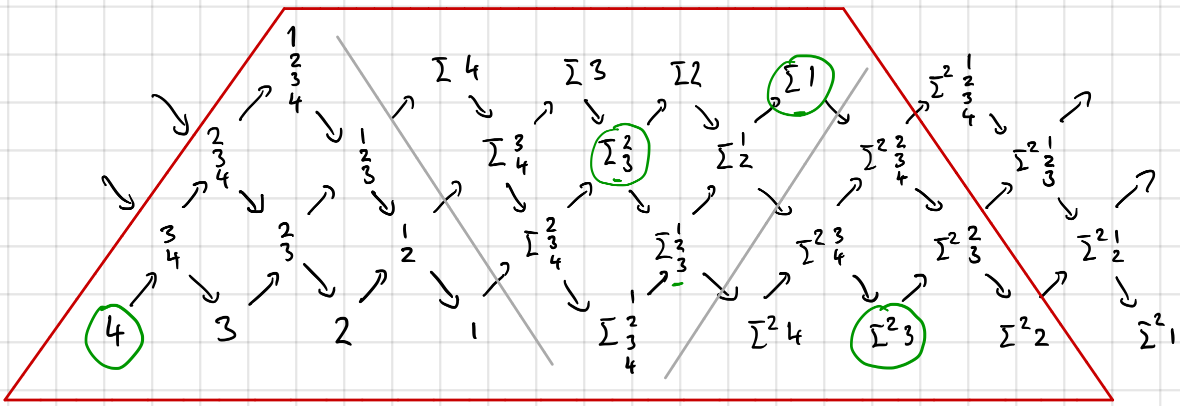


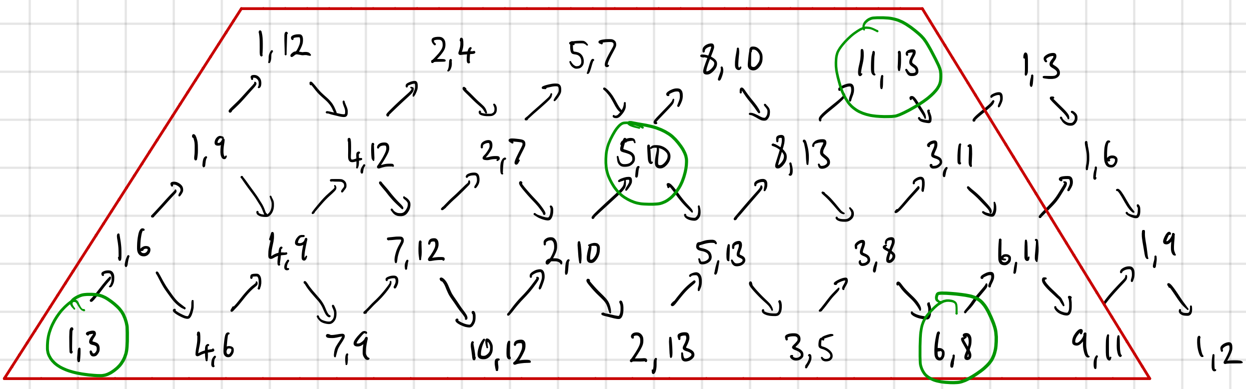
Example 2 (Coelho Simões)

$w=2$, $Q = A_4$ $(1 \to 2 \to 3 \to 4)$, $W_{A_4} \cong S_5$, $R = \{(12), (23), (34), (45)\}$.

$c = (12)(23)(34)(45) = (1\,2\,3\,4\,5)$

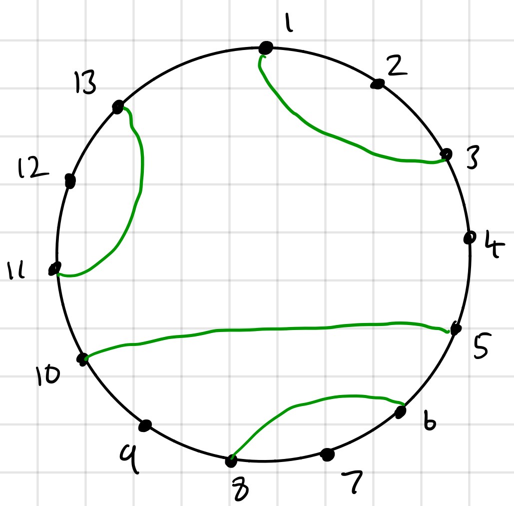
Consider $c = \underbrace{t_3}_{u_1} \underbrace{t_{123}\, t_1}_{u_2} \underbrace{t_4}_{u_3} = \underbrace{(34)}_{u_1} \underbrace{(14)(12)}_{u_2} \underbrace{(45)}_{u_3}$





$NC_+^w(A_n) = NC^w_{\text{pairs}}\big((w+1)(n+1)-2\big)$ = maximal collections of $(w+1)$-diagonals

(= diagonals splitting P into polygons with multiples of $w+1$ vertices) of a $(w+1)(n+1)-2$-gon P.

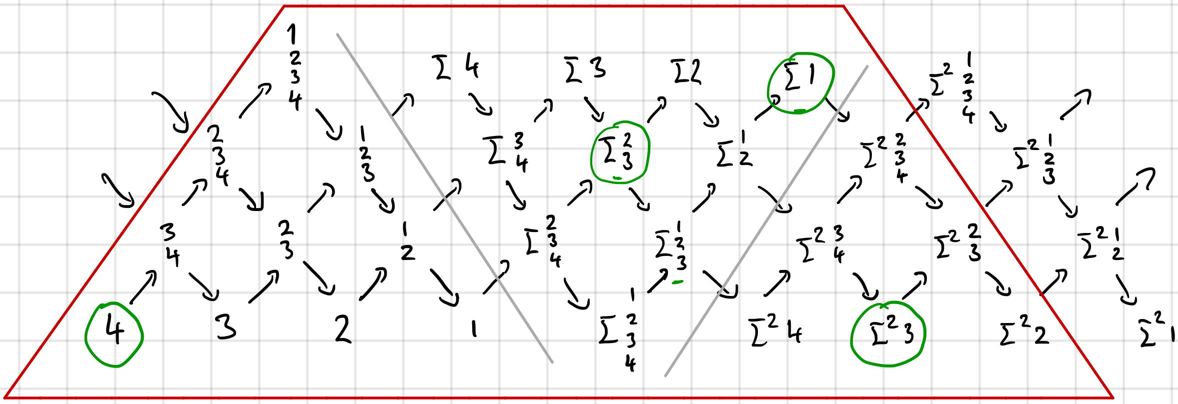


Example 2

$w = 2, \quad Q = A_4 \ (1 \to 2 \to 3 \to 4), \quad W_{A_4} \cong S_5, \quad R = \{(12), (23), (34), (45)\}.$

$c = (12)(23)(34)(45) = (1\,2\,3\,4\,5)$

Consider $c = \underbrace{t_3}_{u_1} \underbrace{t_{123}\, t_1}_{u_2} \underbrace{t_4}_{u_3} = \underbrace{(34)}_{u_1} \underbrace{(14)(12)}_{u_2} \underbrace{(45)}_{u_3}$



Noncrossing? (Armstrong, Krattenthaler - Stump)

$NC^w(A_n) \cong NC^w(n+1)$ = noncrossing partitions of $w(n+1)$ whose blocks have sizes that are multiples of w.

$NC^w_+(A_n) \cong NC^w_+(n+1)$ = as above, but with $1, w(n+1)$ in the same block.

Take $c = u_1 u_2 u_3$ and consider $u_1 u_2 = \underbrace{(34)}_{u_1} \underbrace{(14)(12)}_{u_2}$

$u_i \longmapsto w \cdot u_i - (w+1-i)$ $\qquad u_1 = (34) \longmapsto (68) \longmapsto (4\,6)$ $\qquad (1\,3\,7)$

$\qquad\qquad u_2 = (14)(12) \longmapsto (28)(24) \longmapsto (1\,7)(1\,3)$